

DECOUPLINGS FOR d -DIMENSIONAL SURFACES IN \mathbb{R}^{2d}

CHANGKEUN OH

ABSTRACT. Bourgain and Demeter obtained the sharp l^p decoupling for two-dimensional nondegenerate surfaces in \mathbb{R}^4 . As a generalization of their results, we study the l^p decoupling for d -dimensional surfaces in \mathbb{R}^{2d} . Especially, we obtain the sharp l^p decoupling for 3-dimensional nondegenerate quadratic surfaces in \mathbb{R}^6 .

1. INTRODUCTION

Let $d \geq 2$. Consider a d -dimensional surface in \mathbb{R}^{2d}

$$S = \{(\xi_1, \dots, \xi_d, \Phi_1(\xi_1, \dots, \xi_d), \dots, \Phi_d(\xi_1, \dots, \xi_d)) : (\xi_1, \dots, \xi_d) \in [0, 1]^d\}.$$

Throughout the paper, we assume that the Jacobian of (Φ_1, \dots, Φ_d) is not identically zero:

$$\begin{vmatrix} \frac{\partial \Phi_1}{\partial \xi_1} & \dots & \frac{\partial \Phi_1}{\partial \xi_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_d}{\partial \xi_1} & \dots & \frac{\partial \Phi_d}{\partial \xi_d} \end{vmatrix} \neq 0.$$

In addition, we assume that the functions Φ_i are homogeneous polynomials of degree two for all i . If the surface S satisfies these conditions, we say that S is a d -dimensional *nondegenerate* surface in \mathbb{R}^{2d} . For simplicity, we define a function $\Phi : [0, 1]^d \rightarrow \mathbb{R}^d$ by $\Phi(\xi_1, \dots, \xi_d) = (\Phi_1(\xi_1, \dots, \xi_d), \dots, \Phi_d(\xi_1, \dots, \xi_d))$. We will use the notation $e(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$.

Given a function $g : [0, 1]^d \rightarrow \mathbb{C}$ and a rectangle $\theta \subset [0, 1]^d$, we define the *extension* operator E_θ associated with the surface S by

$$E_\theta g(x) = \int_\theta g(\xi) e(x_1 \xi_1 + \dots + x_d \xi_d + (x_{d+1}, \dots, x_{2d}) \cdot \Phi(\xi_1, \dots, \xi_d)) d\xi_1 \dots d\xi_d.$$

If $\theta = [0, 1]^d$, then we sometimes write $Ef = E_S f$ instead of $E_{[0, 1]^d} f$.

We will write $B_N = B_N(c_B) = B(c_B, N)$ for the cube $c_B + [0, N]^{2d}$, and we define the weight associated with B_N by

$$w_{B_N}(x) = \frac{1}{(1 + \frac{|x - c_B|}{N})^{100d}}.$$

For $1 \leq p < \infty$, for $g : \mathbb{R}^{2d} \rightarrow [0, \infty)$ and $f : \mathbb{R}^{2d} \rightarrow \mathbb{C}$ we define the weighted integral by $\|f\|_{L^p(g)} = \|fg^{\frac{1}{p}}\|_{L^p(\mathbb{R}^{2d})}$.

For each surface S , and for each hyperplane L in \mathbb{R}^d containing the origin, we define a lower dimensional surface $S|_L$ by $S|_L = \{(\xi_2, \dots, \xi_d, \Phi(R(0, \xi_2, \dots, \xi_d)))\}$, where R is a rotation satisfying that $R^{-1}(L) = \{0\} \times \mathbb{R}^{d-1}$.

Our main result is the following l^p decoupling theorem.

Theorem 1.1. Let S be a d -dimensional nondegenerate surface in \mathbb{R}^{2d} . We assume that the following lower dimensional decouplings hold: for each function $f : [0, 1]^{d-1} \rightarrow \mathbb{C}$, for each hyperplane L in \mathbb{R}^d containing the origin, $p \geq 2$ and $N \geq 1$

$$\|E_{S|_L} f\|_{L^p(w_{B_N}^4)} \leq C_{S,p} N^{\gamma(p)} \left(\sum_{\substack{\theta: l(\theta) = N^{-1/2} \\ \theta \subset \mathbb{R}^{d-1}}} \|E_\theta f\|_{L^p(w_{B_N}^4)}^p \right)^{\frac{1}{p}},$$

where the constant $C_{S,p}$ is independent of the choice of L .

Then we have the following decoupling inequality: if $f : [0, 1]^d \rightarrow \mathbb{C}$, then for each $p \geq 2$ and $N \geq 1$

$$\|E_S f\|_{L^p(w_{B_N})} \leq D(N, p) \left(\sum_{\substack{\theta: l(\theta)=N^{-1/2} \\ \theta \in \mathbb{R}^d}} \|E_\theta f\|_{L^p(w_{B_N})}^p \right)^{\frac{1}{p}},$$

where $D(N, p) \leq C_{S,p,\epsilon} N^\epsilon \max(N^{\gamma(p)}, N^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}, N^{\frac{d}{2}-\frac{2d}{p}})$ for each $\epsilon > 0$.

We only use a sum such as $\sum_{\substack{\theta: l(\theta)=N^{-1/2} \\ \theta \in \mathbb{R}^m}}$ when the reciprocal of $N^{-1/2}$ is an integer, and the sum runs over all cubes J in \mathbb{R}^m of the form $[\frac{j_1}{N^{1/2}}, \frac{j_1+1}{N^{1/2}}] \times \dots \times [\frac{j_m}{N^{1/2}}, \frac{j_m+1}{N^{1/2}}]$ for integers j_1, \dots, j_m . Note that we can assume that the reciprocals of $N^{-1/2}$ and ϵ are dyadic numbers, namely, $N^{-1/2} = 2^{-l}$ and $\epsilon = 2^{-m}$ for some integers l and m . This is because the general case follows from these cases. Thus, throughout this paper, we always assume this. For simplicity, we denote the collection $\{\theta : l(\theta) = N^{-1/2}\}$ by $\mathcal{P}_{N^{-1}}$.

Considering that Bourgain and Demeter's argument uses lower dimensional decouplings, the assumption in Theorem 1.1 is natural. Moreover, if the constant $\gamma(p)$ in Theorem 1.1 is optimal, then the upper bound of the constant $D(N, p)$ is sharp up to N^ϵ losses. Theorem 1.1 says that the decoupling for d -dimensional surface in \mathbb{R}^{2d} can be deduced from the lower dimensional decouplings, but the theorem itself does not give the sharp decoupling because we assumed the lower dimensional decouplings.

Let S be a d -dimensional nondegenerate surface in \mathbb{R}^{2d} , and let k be an integer with $0 \leq k \leq d-1$. We say that S is of *type k* if there exists a k -dimensional subspace V in \mathbb{R}^d such that

$$(1.1) \quad (\xi_1, \dots, \xi_d) \in V \Rightarrow (\Phi_1(\xi_1, \dots, \xi_d), \dots, \Phi_d(\xi_1, \dots, \xi_d)) = 0$$

but there does not exist a $(k+1)$ -dimensional subspace V in \mathbb{R}^d satisfying (1.1). Note that the type of a surface is unique.

Corollary 1.2. Fix integers d, k satisfying $d \geq 2$ and $0 \leq k \leq d-1$. Let S be a d -dimensional surface of type k in \mathbb{R}^{2d} . If $f : [0, 1]^d \rightarrow \mathbb{C}$, then for each $p \geq 2$ and $N \geq 1$

$$\|E_{[0,1]^d} f\|_{L^p(w_{B_N})} \leq D_k(N, p) \left(\sum_{\theta \in \mathcal{P}_{N^{-1}}} \|E_\theta f\|_{L^p(w_{B_N})}^p \right)^{\frac{1}{p}},$$

where

$$D_{d-1}(N, p) \leq C_{p,\epsilon,S} N^\epsilon \max(N^{\frac{d}{2}-\frac{2d}{p}}, N^{\frac{d-1}{2}(1-\frac{2}{p})}),$$

$$D_k(N, p) \leq C_{p,\epsilon,S} N^\epsilon \max(N^{\frac{d}{2}-\frac{2d}{p}}, N^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}, N^{\frac{d-1}{2}-\frac{d}{p}}, N^{(\frac{1}{4}-\frac{1}{2p})(d+k-1)})$$

for $0 \leq k < d-1$ and $\epsilon > 0$. Here, the positive constant $C_{p,\epsilon,S}$ depends on p, ϵ and S but not N .

Considering that the l^2 decoupling constant for the hyperbolic paraboloid depends on the dimension of the largest affine subspace of the hyperbolic paraboloid, the definition of the type seems to be natural. In fact, for $d = 3$, Corollary 1.2 gives the sharp decoupling for any nondegenerate surface. Therefore, we close the l^p decoupling problem for 3-dimensional quadratic surfaces in \mathbb{R}^6 . However, for $d \geq 4$, Corollary 1.2 does not always give the sharp decoupling even though for each k there is a surface of type k such that the inequality in Corollary 1.2 is optimal. For the necessity conditions, see the end of Section 1.

The $d = 2$ case of Corollary 1.2 was established by Bourgain and Demeter [5]. This means that Theorem 1.1 is a generalization of their result to higher dimensions. The decoupling problem was introduced by Wolff [20]. After that, some progress in the l^p decoupling for hypersurfaces has been made in [3, 12, 13, 14, 15, 18]. Recently, Bourgain and Demeter [7] proved the decoupling conjectures for the paraboloid and the cone. Based on their results, the l^p decouplings for surfaces with larger than one have been studied in [4, 5, 6, 9, 10]. Moreover, it is known that their results are closely related to number theory, in particular, the estimates on the Riemann zeta function on the critical line and Parsell-Vinogradov systems. Our result can also be used to obtain the upper bound of the number of some system of Diophantine equations; see [6, 10].

To use Bourgain and Guth's multilinear argument, we have to obtain the decoupling restricted to a zero set of a polynomial. Bourgain and Demeter used lower dimensional decouplings to obtain the decoupling for hypersurfaces restricted to a hyperplane, which is a zero set of a polynomial of degree one. If the codimension of a surface is larger than one, then we have to deal with the decoupling for surfaces restricted to a zero set of a polynomial of degree greater than one. We give a way to overcome this difficulty, which uses properties

of the polynomial and the decoupling inequality but not properties of the surface S . Recently, Bourgain, Demeter and Shaoming [9] obtained the sharp decoupling for 2-dimensional surfaces in \mathbb{R}^9 , but the author believes that the main difficulty in the proof of Theorem 1.1 is different from that of theirs.

Throughout the paper, we write $A \lesssim B$ if $A \leq cB$ and $A \sim B$ if $c^{-1}A \leq B \leq cA$. The constant c will in general depend on fixed parameter p and sometimes on the variable parameter ϵ but not N . If R is a rectangular box and c is a positive real number, then we denote by cR the box obtained by dilating R by a number c about its center. For each $N > 1$, let $\delta = 1/N$. If L is a set in \mathbb{R}^d or \mathbb{R}^{2d} , we denote by L_δ the δ -neighborhood of L in \mathbb{R}^d or \mathbb{R}^{2d} , respectively. For $x = (x_1, \dots, x_{2d}) \in \mathbb{R}^{2d}$, we use the notation $x = (x', x'')$, where $x' = (x_1, \dots, x_d)$ and $x'' = (x_{d+1}, \dots, x_{2d})$.

A standard computation with $f = 1_{[0,1]^d}$ reveals that

$$N^{\frac{d}{2} - \frac{2d}{p}} \lesssim D(N, p).$$

A randomization argument and Proposition 5.1 show that $N^{\frac{d}{2}(\frac{1}{2} - \frac{1}{p})} \lesssim D(N, p)$. To see this, let $\{r_\theta\}_{\theta \in \mathcal{P}_{N-1}}$ be Rademacher functions on $[0, 1]$, and let ψ be a smooth function supported in $[0, 1]^{2d}$. For each $\theta \in \mathcal{P}_{N-1}$, we write $\theta = c_\theta + [-\frac{1}{2N^{1/2}}, \frac{1}{2N^{1/2}}]^d$ and take a function $h_\theta : \mathbb{R}^{2d} \rightarrow \mathbb{C}$ to be $\widehat{h_\theta}(\xi', \xi'') = \psi(10N(\xi' - c_\theta, \xi'' - \Phi(c_\theta)))$ so that $\text{supp}(\widehat{h_\theta}) \subset (c_\theta, \Phi(c_\theta)) + [0, \frac{1}{10N}]^d$. Define $h_{\theta,t}(x) = r_\theta(t)h_\theta(x)$ for $t \in [0, 1]$. From the decoupling and Proposition 5.1, we have

$$\left(\int_0^1 \left\| \sum_\theta h_{\theta,t}(x) \right\|_{L_x^p}^p dt \right)^{1/p} \lesssim D(N, p) \left(\sum_\theta \|h_\theta\|_p^p \right)^{1/p}.$$

Now by using Fubini's theorem and Khinchin's inequality, we obtain

$$\left\| \left(\sum_\theta |h_\theta|^2 \right)^{1/2} \right\|_p \lesssim \left(\int_0^1 \left\| \sum_\theta h_{\theta,t}(x) \right\|_{L_x^p}^p dt \right)^{1/p}.$$

These two inequalities lead to $N^{\frac{d}{4}} \lesssim D(N, p)N^{\frac{d}{2p}}$. Hence, we obtain $N^{\frac{d}{2}(\frac{1}{2} - \frac{1}{p})} \lesssim D(N, p)$. Therefore, Theorem 1.1 is optimal up to N^ϵ losses.

Next, suppose that S is of type $d-1$. We take a function $g = 1_{V_\delta}$, where V is a $(d-1)$ -dimensional subspace satisfying the condition (1.1). This function gives the bound

$$N^{\frac{d-1}{2}(1 - \frac{2}{p})} \lesssim D_{d-1}(N, p).$$

Moreover, if $d = 3$ and $k = 0$ or 1 , then

$$\max(N^{\frac{d}{2} - \frac{2d}{p}}, N^{\frac{d}{2}(\frac{1}{2} - \frac{1}{p})}, N^{\frac{d-1}{2} - \frac{d}{p}}, N^{(\frac{1}{4} - \frac{1}{2p})(d+k-1)}) = \max(N^{\frac{d}{2} - \frac{2d}{p}}, N^{\frac{d}{2}(\frac{1}{2} - \frac{1}{p})}).$$

Lastly, let $d \geq 4$ and $0 \leq k \leq d-2$. We point out that there exists a d -dimensional surface of type k in \mathbb{R}^{2d} such that

$$\max(N^{\frac{d}{2} - \frac{2d}{p}}, N^{\frac{d}{2}(\frac{1}{2} - \frac{1}{p})}, N^{\frac{d-1}{2} - \frac{d}{p}}, N^{(\frac{1}{4} - \frac{1}{2p})(d+k-1)}) \lesssim D_k(N, p).$$

We define the surface S_k by

$$S_k = \{(\xi_1, \dots, \xi_d, \xi_1^2 + \dots + \xi_{d-k}^2, \xi_1 \xi_2, \dots, \xi_1 \xi_d) : (\xi_1, \dots, \xi_d) \in [0, 1]^d\}.$$

Let $f : [0, 1]^{d-k-1} \rightarrow \mathbb{C}$ be a function, and take the function $g(\xi) = 1_{[0,\delta]}(\xi_1)f(\xi_2, \dots, \xi_{d-k})$. This gives the bound

$$\max(N^{\frac{d-1}{2} - \frac{d}{p}}, N^{(\frac{1}{4} - \frac{1}{2p})(d+k-1)}) \lesssim D_k(N, p)$$

because of the necessary condition of the decoupling for the hyperbolic paraboloid.

1.1. Outline of the paper. In Section 2, we define a transversality and obtain the bilinear Kekeya inequality. In Section 3, we give some definitions, and we get some lemmas such as parabolic rescaling. Section 4 is the most important section in this paper. In this section, we study relations between the linear l^p decoupling and the bilinear l^p decoupling. In Section 5, we give well-known equivalent formulations of the decoupling problem. In Section 6, we review a standard wave packet decomposition. In Section 7, we complete the proof of Theorem 1.1. In Section 8, we deduce Corollary 1.2.

2. TRANSVERSALITY

In this section, we will study a transversality condition. More precisely, we will define some concepts related to the transversality condition, and then we will obtain the bilinear Keakeya inequality. Suppose that S is a d -dimensional nondegenerate surfaces in \mathbb{R}^{2d} .

2.1. Definitions. We can take d linearly independent normal vectors to S at $(p, \Phi(p))$:

$$\begin{aligned} m_1(p) &= \left(\frac{\partial \Phi_1}{\partial \xi_1}(p), \dots, \frac{\partial \Phi_1}{\partial \xi_d}(p), -1, 0, \dots, 0, 0 \right), \\ m_2(p) &= \left(\frac{\partial \Phi_2}{\partial \xi_1}(p), \dots, \frac{\partial \Phi_2}{\partial \xi_d}(p), 0, -1, \dots, 0, 0 \right), \\ &\vdots \\ m_d(p) &= \left(\frac{\partial \Phi_d}{\partial \xi_1}(p), \dots, \frac{\partial \Phi_d}{\partial \xi_d}(p), 0, 0, \dots, 0, -1 \right). \end{aligned}$$

Fix $\nu > 0$. We say that two points p_1, p_2 in \mathbb{R}^d are ν -transverse if

$$(2.1) \quad J(p_1, p_2) = |\det(m_1(p_1), \dots, m_d(p_1), m_1(p_2), \dots, m_d(p_2))| > \nu.$$

Note that $J(p_1, p_2) = J(p_1 - p_2, 0)$. This symmetry makes our proof easier.

We denote by $\pi : S \rightarrow \mathbb{R}^d$ the projection map deleting the $(d+1), \dots, 2d$ coordinates. We say that two sets $E_1, E_2 \subset \mathbb{R}^d$ are ν -transverse if any two points $p_1 \in E_1$ and $p_2 \in E_2$ are ν -transverse.

2.2. The bilinear Keakeya inequality.

Lemma 2.1 (The bilinear restriction theorem). Let $\nu > 0$. Let R_1, R_2 be cubes in $[0, 1]^d$, which are ν -transverse. Then for each $g_i : R_i \rightarrow \mathbb{C}$, we have

$$|||E_{R_1} g_1 E_{R_2} g_2|||_{L^4(\mathbb{R}^{2d})}^{\frac{1}{2}} \lesssim_{\nu} (\|g_1\|_{L^2(R_1)} \|g_2\|_{L^2(R_2)})^{\frac{1}{2}}.$$

Proof. We will use the change of variables

$$(\xi_1, \dots, \xi_d, \eta_1, \dots, \eta_d) \mapsto (\xi_1 + \eta_1, \dots, \xi_d + \eta_d, \Phi(\xi_1, \dots, \xi_d) + \Phi(\eta_1, \dots, \eta_d)).$$

Note that the Jacobian of this mapping is given by $J(\xi_1, \dots, \xi_d, \eta_1, \dots, \eta_d)$, which was defined in (2.1). Since this transformation is defined in terms of homogeneous polynomials and $J(\xi, \eta) \neq 0$ for $\xi \in R_1$ and $\eta \in R_2$, it follows from Bezout's theorem that it has a uniformly bounded multiplicity. Hence

$$|E_{R_1} g_1(x) E_{R_2} g_2(x)| = \widehat{F J^{-1}}(x),$$

where $F = g_1 g_2$ and $|J^{-1}(u)| < \nu^{-1}$. By using Plancherel's identity, we have

$$|||E_{R_1} g_1 E_{R_2} g_2|||_{L^4}^{\frac{1}{2}} = \|F J^{-1}\|_{L^2}^{\frac{1}{2}} \lesssim_{\nu} \left(\int |F^2(u) J^{-1}(u)| du \right)^{\frac{1}{4}} = (\|g_1\|_2 \|g_2\|_2)^{\frac{1}{2}}.$$

This completes the proof of Lemma 2.1. \square

Let \mathcal{P} be a collection of all cubes P_p on a d -dimensional affine subspaces in \mathbb{R}^{2d} with a point $p \in [0, 1]^d$ satisfying the following: the side lengths of each P_p are equal to $N^{1/2}$ and the axes of P_p span a subspace spanned by d -vectors $m_1(p), \dots, m_d(p)$. We will say that P_p is *associated with* p .

Definition 2.2. Let $\nu > 0$. We say that two families \mathcal{P}_i are ν -transverse if there exist two cubes α_1, α_2 , which are ν -transverse, such that each $P_p \in \mathcal{P}_i$ is associated with a point $p \in \alpha_i$.

Suppose $P_{j,a}$ are elements of \mathcal{P}_j , where $1 \leq a \leq N_j$ and $j = 1, 2$. We denote by $\tilde{P}_{j,a}$ the 1-neighborhood of $P_{j,a}$ in \mathbb{R}^{2d} , and denote by $T_{j,a}$ the characteristic function of $\tilde{P}_{j,a}$.

Proposition 2.3 (The bilinear Keakeya-type inequality). Let $\nu > 0$. Assume that two families $\mathcal{P}_j = \{P_{j,a} : 1 \leq a \leq N_j\}$, $j = 1, 2$, are ν -transverse. Then we have

$$\int_{\mathbb{R}^{2d}} \sum_{a=1}^{N_1} \sum_{b=1}^{N_2} T_{1,a}(x) T_{2,b}(x) dx \lesssim_{\nu} N_1 N_2.$$

Proof. Fix an element $P_{i,a} \in \mathcal{P}_i$ and a point $v_{i,a} \in \mathbb{R}^{2d}$. Suppose that $P_{i,a}$ is associated with $p_{i,a}$. Let $w_{i,a} = p_{i,a} + [0, \delta^{1/2}]^d \subset [0, 1]^d$. Define $f_{w_{i,a}} : [0, 1]^d \rightarrow \mathbb{C}$ by $f_{w_{i,a}}(\xi) = e(-v_{i,a} \cdot (\xi, \Phi(\xi))) \chi_{w_{i,a}}(\xi)$. Then

$$|Ef_{w_{i,a}}(x)| = \left| \int_{w_{i,a}} e((x - v_{i,a}) \cdot ((\xi, \Phi(\xi)) - (p_{i,a}, \Phi(p_{i,a})))) d\xi \right|.$$

Let $\tilde{P}_{i,a}(v_{i,a})$ be the translation of the rectangle $cN^{1/2}\tilde{P}_{i,a}$ centered at $v_{i,a}$ for small constant $c > 0$. Then for any $x \in \tilde{P}_{i,a}(v_{i,a})$

$$|Ef_{w_{i,a}}(x)| \geq \int_{w_{i,a}} \frac{1}{1000} d\xi = \frac{\delta^{d/2}}{1000}.$$

Hence, we get $|Ef_{w_{i,a}}(x)| \geq \frac{1}{10^3} \delta^{d/2} \chi_{\tilde{P}_{i,a}(v_{i,a})}(x)$, so

$$|Ef_{w_{1,a}}(x) Ef_{w_{2,b}}(x)| \geq \frac{\delta^d}{10^6} \chi_{\tilde{P}_{1,a}(v_{1,a})}(x) \chi_{\tilde{P}_{2,b}(v_{2,b})}(x)$$

for any transverse sets $w_{1,a}$ and $w_{2,b}$, for any points $v_{1,a}, v_{2,b} \in \mathbb{R}^{2d}$ and for any $x \in \mathbb{R}^{2d}$. By the bilinear restriction theorem, we get

$$\begin{aligned} \delta^d N_1 N_2 &\gtrsim \left\| \left(\sum_{a=1}^{N_1} |f_{w_{1,a}}|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^d)}^2 \left\| \left(\sum_{b=1}^{N_2} |f_{w_{2,b}}|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^d)}^2 \\ &\gtrsim \left\| \left(\sum_{a,b} |Ef_{w_{1,a}}(x) Ef_{w_{2,b}}(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^{2d})}^2 \gtrsim \delta^{2d} \int_{\mathbb{R}^{2d}} \sum_{a,b} \chi_{\tilde{P}_{1,a}(v_{1,a})}(x) \chi_{\tilde{P}_{2,b}(v_{2,b})}(x) dx. \end{aligned}$$

Next, we use the change of variables $y = c\delta^{1/2}x$. This gives the desired results. \square

By interpolating two points $p = 4$ and $p = \infty$ via Hölder's inequality, we obtain

$$\int_{\mathbb{R}^{2d}} \left| \sum_{a=1}^{N_1} \sum_{b=1}^{N_2} T_{1,a}(x) T_{2,b}(x) \right|^{\frac{p}{4}} dx \lesssim_{\nu} (N_1 N_2)^{\frac{p}{4}}$$

for any $4 \leq p < \infty$. Moreover, a standard argument gives

Corollary 2.4. Let $\nu > 0$. Assume that two families $\mathcal{P}_j = \{P_{j,a} : 1 \leq a \leq N_j\}$, $j = 1, 2$, are ν -transverse. Then for any $4 \leq p < \infty$, we have

$$\int_{\mathbb{R}^{2d}} \prod_{i=1}^2 \left| \sum_{a=1}^{N_i} T_{i,a} * g_{i,a}(x) \right|^{\frac{p}{4}} dx \lesssim_{\nu} \prod_{i=1}^2 \left(\sum_{a=1}^{N_i} \|g_{i,a}\|_{L^1(\mathbb{R}^{2d})} \right)^{\frac{p}{4}}.$$

for all nonnegative functions $g_{i,a} \in L^1(\mathbb{R}^{2d})$.

Proof. First, observe that

$$(2.2) \quad \int_{\mathbb{R}^{2d}} \left| \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} u_a v_b T_{1,a}(x) T_{2,b}(x) \right|^{\frac{p}{4}} dx \lesssim_{\nu} \left(\sum_{a=1}^{\infty} \sum_{b=1}^{\infty} u_a v_b \right)^{\frac{p}{4}}$$

for all non-negative real numbers u_a, v_b . Let $c_{i,a}$ be the center of $\tilde{P}_{i,a}$. Next, we take a finitely overlapping cover of \mathbb{R}^{2n} by translating a fixed tube $\tilde{P}_{i,a}$, and call this cover \mathcal{G} . Then

$$T_{i,a} * g_{i,a}(x) = \int_{x - \tilde{P}_{i,a}} g_{i,a}(y) dy \leq \sum_{P \in \mathcal{G}} (\tilde{T}_{i,a,P}(x) \int_P g_{i,a}(y) dy),$$

where $\tilde{T}_{i,a,P}$ is a characteristic function of $100P + c_{i,a}$. With this and (2.2), we get the desired results. \square

3. SOME DEFINITIONS AND LEMMAS

Let S be a d -dimensional nondegenerate surface in \mathbb{R}^{2d} . We define two constants. For any $2 \leq p < \infty$, $1 \leq l < \infty$ and any dyadic number $N^{1/2} \geq 1$, we denote by $D(N, p, l)$ the smallest constant such that the following decoupling holds;

$$\|E_S g\|_{L^p(w_{B_N}^l)} \leq D(N, p, l) \left(\sum_{\theta \in \mathcal{P}_\delta} \|E_\theta g\|_{L^p(w_{B_N}^l)}^p \right)^{\frac{1}{p}}$$

for each $g : [0, 1]^d \rightarrow \mathbb{C}$. If $l = 1$, we sometimes write $D(N, p)$ instead of $D(N, p, 1)$.

Fix $\nu > 0$. We denote by $D_{bil}(N, p, \nu)$ the smallest constant such that the bilinear decoupling holds;

$$\| |E_{R_1} g_1 E_{R_2} g_2|^{\frac{1}{2}} \|_{L^p(w_{B_N}^2)} \leq D_{bil}(N, p, \nu) \left(\prod_{i=1}^2 \sum_{\theta \in \mathcal{P}_\delta} \|E_\theta g_i\|_{L^p(w_{B_N}^2)}^p \right)^{\frac{1}{2p}}$$

for any functions $g_1, g_2 : [0, 1]^d \rightarrow \mathbb{C}$ and dyadic cubes R_1, R_2 , which are ν -transverse. Note that $D_{bil}(N, p, \nu) \leq D(N, p, 2)$.

Throughout this paper, we will use the following localization lemma frequently. This lemma is identical to Lemma 7.1 in [10].

Lemma 3.1 (The localization principle). Let \mathcal{W} be the collection of positive integrable functions on \mathbb{R}^{2d} . Let $O_1, O_2 : \mathcal{W} \rightarrow [0, \infty]$ have the following four properties:

- (1) $O_1(1_B) \lesssim O_2(w_B)$ for all cubes $B \subset \mathbb{R}^{2d}$ of side length R
- (2) $O_1(u + v) \leq O_1(u) + O_1(v)$, for each $u, v \in \mathcal{W}$
- (3) $O_2(u + v) \geq O_2(u) + O_2(v)$, for each $u, v \in \mathcal{W}$
- (4) If $u \leq v$ then $O_i(u) \leq O_i(v)$.

Then

$$O_1(w_B) \lesssim O_2(w_B)$$

for each cube B with side length R . The implicit constant is independent of R, B and depends on the implicit constant from (1).

As a direct application of Lemma 3.1, one may see that $D(N, p, 1) \lesssim_l D(N, p, l)$ for all $l \geq 1$. We will show the reverse inequality $D(N, p, l) \lesssim_l D(N, p, 1)$ for all $l \geq 1$ at the end of Section 5. Hence, we assume these inequalities in Section 3 and 4.

One of the key propositions is the parabolic rescaling. The proof of Proposition 3.2 is identical to that of Proposition 7.1 in [6].

Proposition 3.2 (Parabolic rescaling). Suppose that two numbers δ, σ with $0 < \delta \leq \sigma$ are dyadic numbers, and let $\tau = a + [0, \sigma^{1/2}]^d \in \mathcal{P}_\sigma$. Then for each $f : [0, 1]^d \rightarrow \mathbb{C}$, we have

$$\|E_\tau f\|_{L^p(w_{B_N})} \lesssim D\left(\frac{\sigma}{\delta}, p\right) \left(\sum_{\theta \in \mathcal{P}_\delta, \theta \subset \tau} \|E_\theta f\|_{L^p(w_{B_N})}^p \right)^{\frac{1}{p}}.$$

Proof. By lemma 3.1, it suffices to show that

$$\|E_\tau f\|_{L^p(B_N)} \lesssim D\left(\frac{\sigma}{\delta}, p\right) \left(\sum_{\theta \in \mathcal{P}_\delta, \theta \subset \tau} \|E_\theta f\|_{L^p(w_{B_N})}^p \right)^{\frac{1}{p}}.$$

We write $a = (a_1, \dots, a_d)$ and define an affine transformation associated with τ by

$$(3.1) \quad L_\tau(\xi_1, \dots, \xi_d) = \left(\frac{\xi_1 - a_1}{\sigma^{1/2}}, \dots, \frac{\xi_d - a_d}{\sigma^{1/2}} \right)$$

so that the image of τ under L_τ is $[0, 1]^d$. Define $g(\xi) = f(L_\tau^{-1}\xi)\sigma^{\frac{d}{2} - \frac{3d}{2p}}$. Through routine calculations, we can see

$$\sigma^{\frac{3d}{2p}} |E_{[0,1]^d} g(\sigma^{1/2}(x' + A_\tau x''), \sigma x'')| = |E_\tau f(x)|,$$

where $x' = (x_1, \dots, x_d)$, $x'' = (x_{d+1}, \dots, x_{2d})$ and A_τ is some $d \times d$ matrix. We define a linear transformation $M : x \mapsto \bar{x}$ to be $\sigma^{\frac{3d}{2p}} |E_{[0,1]^d} g(\bar{x})| = |E_\tau f(x)|$. Note that the image of B_N under the transformation M is a cylinder C_N , which has dimensions $\sigma^{1/2} N \times \dots \times \sigma^{1/2} N \times \sigma N \times \dots \times \sigma N$. Note also that for any $x \in \mathbb{R}^{2d}$

$$1_{C_N}(x) \lesssim \sum_{B_{\sigma N} \cap C_N \neq \emptyset} w_{B_{\sigma N}}^2(x) \lesssim w_{B_N}(M^{-1}x).$$

Hence, by using a change of variables and changing back to the original variables, we get

$$\begin{aligned} \|E_\tau f\|_{L^p(B_N)}^p &= \|Eg\|_{L^p(C_N)}^p \leq \sum_{B_{\sigma N} \cap C_N \neq \emptyset} \|Eg\|_{L^p(B_{\sigma N})}^p \\ &\lesssim D\left(\frac{\sigma}{\delta}, p\right)^p \sum_{\theta' \in \mathcal{P}_{\delta/\sigma}} \|E_{\theta'} g\|_{L^p(\sum_{B_{\sigma N} \cap C_N \neq \emptyset} w_{B_{\sigma N}}^2)}^p \\ &\lesssim D\left(\frac{\sigma}{\delta}, p\right)^p \sum_{\theta \in \mathcal{P}_\delta, \theta \subset \tau} \|E_\theta f\|_{L^p(w_{B_N})}^p. \end{aligned}$$

This completes the proof of Proposition 3.2. \square

We note an easy lemma. The lemma follows by interpolating L^2 and L^∞ estimates.

Lemma 3.3 (The trivial l^p decoupling). Suppose that rectangles $\theta_1, \dots, \theta_K$ in \mathbb{R}^d are disjoint each other. Then for each $p \geq 2$, $g : [0, 1]^d \rightarrow \mathbb{C}$ and $K \geq 1$

$$\left\| \sum_{i=1}^K E_{\theta_i} g \right\|_p^p \lesssim K^{p-2} \sum_{i=1}^K \|E_{\theta_i} g\|_p^p.$$

Note that Hölder's inequality gives

$$D(N, p) \lesssim N^{\frac{d}{2}(1-\frac{1}{p})}.$$

Now, let $\gamma_{lin}(p)$ be the unique number such that

$$\begin{aligned} \lim_{N \rightarrow \infty} D(N, p) N^{-\gamma_{lin}(p)-\epsilon} &= 0, \text{ for each } \epsilon > 0, \\ \limsup_{N \rightarrow \infty} D(N, p) N^{-\gamma_{lin}(p)+\epsilon} &= \infty, \text{ for each } \epsilon > 0. \end{aligned}$$

Similarly, let $\gamma_{bil}(p)$ be the unique number such that

$$\begin{aligned} \lim_{N \rightarrow \infty} D_{bil}(N, p) N^{-\gamma_{bil}(p)-\epsilon} &= 0, \text{ for each } \epsilon > 0, \\ \limsup_{N \rightarrow \infty} D_{bil}(N, p) N^{-\gamma_{bil}(p)+\epsilon} &= \infty, \text{ for each } \epsilon > 0. \end{aligned}$$

4. LINEAR VERSUS BILINEAR DECOUPLING FOR SURFACES

The goal of this section is to prove Theorem 4.1, which means that the linear l^p decoupling follows from the bilinear l^p decoupling. Since the bilinear l^p decoupling is much easier than the linear l^p decoupling, the theorem is useful.

Let S be a d -dimensional nondegenerate surface in \mathbb{R}^{2d} . For simplicity, we define A_p by $A_p(x) = \max(x^{\gamma(p)}, x^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})})$. Note that $A_p(x)A_p(y) = A_p(xy)$ and $A_p(x)A_p(y)^{-1} = A_p(xy^{-1})$ for any $x \geq y \geq 1$.

Theorem 4.1. For each $\epsilon > 0$ and $p \geq 2$, there exist $C_{p,\epsilon} > 0$ and $\nu > 0$ such that for each $N \geq 1$

$$D(N, p) \leq C_{p,\epsilon} N^\epsilon \sup_{1 < M < N} A_p\left(\frac{N}{M}\right) D_{bil}(M, p, \nu).$$

Observe that Theorem 4.1 implies

$$D(N, p) \leq C'_{p,\epsilon} N^{2\epsilon} \max(A_p(N), N^{\gamma_{bil}(p)}).$$

If $D(N, p) \leq C'_{p,\epsilon} N^{2\epsilon} A_p(N)$, then we obtain Theorem 1.1. Therefore, from this, Theorem 4.1 and the definition of γ_{lin} and γ_{bil} , we can assume that

$$(4.1) \quad \gamma_{lin}(p) \leq \gamma_{bil}(p).$$

We note that if there exists a hyperplane L in \mathbb{R}^d such that $\Phi(L) = 0$, then we can assume that

$$S = \{(\xi_1, \dots, \xi_d, \xi_1^2, 2\xi_1\xi_2, \dots, 2\xi_1\xi_d) : (\xi_1, \dots, \xi_d) \in [0, 1]^d\}$$

by using a change of variables. Moreover, in this case, we have

$$(4.2) \quad J(p_1, p_2) = |\det(m_1(p_1), \dots, m_d(p_1), m_1(p_2), \dots, m_d(p_2))| = 2^d |\xi_1 - \eta_1|$$

for any $p_1 = (\xi_1, \dots, \xi_d)$ and $p_2 = (\eta_1, \dots, \eta_d)$.

4.1. Linear vs bilinear decoupling for surfaces. Define a set $Z = \{(\xi_1, \dots, \xi_d) : J((\xi_1, \dots, \xi_d), 0) = 0\}$. Note that if there exists a hyperplane L in \mathbb{R}^d such that $\Phi(L) = 0$, then Z is a hyperplane in \mathbb{R}^d by (4.2). We define $F(\xi_1, \dots, \xi_d) = J((\xi_1, \dots, \xi_d), 0)$. Note that F is a nonzero polynomial of degree d .

To prove Theorem 4.1, we will prove the following inequality first.

Proposition 4.2. Fix $\bar{\epsilon}, \epsilon > 0$ and $p \geq 2$. There exist dyadic numbers K_0, \dots, K_{d-1} such that $\bar{\epsilon} \leq K_{d-1} \leq \dots \leq K_0$ and for any $f : [0, 1]^d \rightarrow \mathbb{C}$ and $N \geq K_0$

$$\begin{aligned} \|Ef\|_{L^p(B_N)}^p &\leq \sum_{i=0}^{d-1} C_{p,\epsilon} K_i^{C_\epsilon} A_p(K_i^{t_i})^p \sum_{\alpha \in \mathcal{P}_{K_i^{-t_i}}} \|E_\alpha f\|_{L^p(w_{B_N}^2)}^p \\ &\quad + C_{p,\epsilon} K_0^{C_\epsilon} A_p(K_0)^p \sum_{\beta \in \mathcal{P}_{K_0^{-1}}} \|E_\beta f\|_{L^p(w_{B_N}^2)}^p \\ &\quad + C_p K_0^{2pd} D_{bil}(N, p, C_{K_0})^p \sum_{\theta \in \mathcal{P}_\delta} \|E_\theta f\|_{L^p(w_{B_N}^2)}^p, \end{aligned}$$

where t_i is some number satisfying $\frac{1}{4} \leq t_i \leq 1$ and $C_{p,\epsilon}, C_p$ are independent of $\bar{\epsilon}$. Here, $K_i^{t_i/2}$ are dyadic numbers.

Proof of Proposition 4.2. Due to translation invariance, we can assume that $B_N = [0, N]^{2d}$. We will follow the standard formalism in [11]. Fix a cube $B_{K_0^{1/2}}(a)$ in B_N . We take a Schwartz function η on \mathbb{R}^{2d} , with $\hat{\eta}(x) = 1$ on $[-2, 2]^{2d}$ and $\hat{\eta}(x) = 0$ outside $[-4, 4]^{2d}$. We also take the function $\zeta_{B_{K_0^{1/2}}(a)}(x) = K_0^{-d} w_{B_{K_0^{1/2}}(a)}^{100}(x)$ so that $\|\zeta_{B_{K_0^{1/2}}(a)}\|_{L^1(\mathbb{R}^{2d})} \sim 1$. If $a = 0$, we sometimes write $\zeta_{K_0^{1/2}}$ instead of $\zeta_{B_{K_0^{1/2}}(0)}$. For each cube $\alpha = b_\alpha + [0, K_0^{-1/2}]^d \in \mathcal{P}_{K_0^{-1}}$ with some $b_\alpha \in [0, 1]^d$, we define

$$\eta_{K_0^{1/2}, \alpha}(x) = K_0^{-d} e(x \cdot (b_\alpha, \Phi(b_\alpha))) \eta\left(\frac{x}{K_0^{1/2}}\right).$$

By an application of Young's inequality

$$\begin{aligned} \|E_\alpha f(x - \cdot) \eta_{K_0^{1/2}, \alpha}(\cdot)\|_{L^1} &\leq \|E_\alpha f(x - \cdot) \eta_{K_0^{1/2}, \alpha}(\cdot)\|_{L^\infty}^{\frac{1}{2}} \|E_\alpha f(x - \cdot) \eta_{K_0^{1/2}, \alpha}(\cdot)\|_{L^{\frac{1}{2}}}^{\frac{1}{2}} \\ &\lesssim K_0^{-\frac{d}{2}} \|E_\alpha f(x - \cdot) \eta_{K_0^{1/2}, \alpha}(\cdot)\|_{L^1}^{\frac{1}{2}} \|E_\alpha f(x - \cdot) \eta_{K_0^{1/2}, \alpha}(\cdot)\|_{L^{\frac{1}{2}}}^{\frac{1}{2}}. \end{aligned}$$

Hence, we have

$$|E_\alpha f(x)| = |E_\alpha f * \eta_{K_0^{1/2}, \alpha}(x)| \lesssim \left(\int_{\mathbb{R}^{2d}} |E_\alpha f(x - y)|^{\frac{1}{2}} \frac{1}{K_0^d} \left| \eta\left(\frac{y}{K_0^{1/2}}\right) \right|^{\frac{1}{2}} dy \right)^2.$$

Define

$$c_\alpha(B_{K_0^{1/2}}(a)) = \left(\int_{\mathbb{R}^{2d}} |E_\alpha f(y)|^{\frac{1}{2}} \zeta_{B_{K_0^{1/2}}(a)}(y) dy \right)^2.$$

Note that for any $x \in B_{K_0^{1/2}}(a)$

$$\begin{aligned} |E_\alpha f(x)| &\lesssim \left(\int_{\mathbb{R}^{2d}} |E_\alpha f(y)|^{\frac{1}{2}} \zeta_{B_{K_0^{1/2}}(0)}(x - y) dy \right)^2 \lesssim c_\alpha(B_{K_0^{1/2}}(a)) \\ &\lesssim \int |E_\alpha f(y)| \zeta_{B_{K_0^{1/2}}(a)}(y) dy \lesssim \int_{\mathbb{R}^{2d}} |E_\alpha f(x - y)| \zeta_{B_{K_0^{1/2}}(0)}(y) dy. \end{aligned}$$

Let $\alpha^* \in \mathcal{P}_{K_0^{-1}}$ be a cube maximizing the value $c_\alpha(B_{K_0^{1/2}}(a))$. There are two possibilities.

(Case 1: a transverse case) Consider the case that there is some cube $\alpha^{**} \in \mathcal{P}_{K_0^{-1}}$ such that $\alpha^{**} \cap (Z_{5\sqrt{d}K_0^{-1/2}} + b_{\alpha^*}) = \phi$ (Hence, α^* and α^{**} are C_{K_0} -transverse for some small $C_{K_0} > 0$) and $c_{\alpha^{**}}(B_{K_0^{1/2}}(a)) \geq K_0^{-d/2} c_{\alpha^*}(B_{K_0^{1/2}}(a))$. Then for any $x \in B_{K_0^{1/2}}(a)$ we have

$$|Ef(x)| = \left| \sum_{\alpha} E_{\alpha} f(x) \right| \leq K_0^{\frac{3d}{4}} (c_{\alpha^{**}}(B_{K_0^{1/2}}(a)) c_{\alpha^*}(B_{K_0^{1/2}}(a)))^{1/2},$$

and we also have

$$\begin{aligned} & |c_{\alpha^*}(B_{K_0^{1/2}}(a)) c_{\alpha^{**}}(B_{K_0^{1/2}}(a))|^{\frac{1}{2}} \\ & \lesssim \iint_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} |E_{\alpha^*} f(x - y_1) E_{\alpha^{**}} f(x - y_2)|^{\frac{1}{2}} \zeta_{K_0^{1/2}}(y_1) \zeta_{K_0^{1/2}}(y_2) dy_1 dy_2. \end{aligned}$$

Raising to the p power, integrating on the cube $B_{K_0^{1/2}}(a)$ and Hölder's inequality give

$$\begin{aligned} & \|Ef\|_{L^p(B_{K_0^{1/2}}(a))}^p \\ & \lesssim K_0^{pd} \iint_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \| |E_{\alpha^*} f(\cdot - y_1) E_{\alpha^{**}} f(\cdot - y_2)|^{\frac{1}{2}} \|_{L^p(B_{K_0^{1/2}}(a))}^p \zeta_{K_0^{1/2}}(y_1) \zeta_{K_0^{1/2}}(y_2) dy_1 dy_2 \\ & \lesssim K_0^{pd} \iint \sum_{\substack{\alpha, \beta \in \mathcal{P}_{K_0^{-1}} \\ (\alpha, \beta): C_{K_0} \text{-trans}}} \| |E_{\alpha} f(\cdot - y_1) E_{\beta} f(\cdot - y_2)|^{\frac{1}{2}} \|_{L^p(B_{K_0^{1/2}}(a))}^p \zeta_{K_0^{1/2}}(y_1) \zeta_{K_0^{1/2}}(y_2) dy_1 dy_2. \end{aligned}$$

(Case 2: a non-transverse case) Suppose that we are not in Case 1. Then if a cube $\alpha \in \mathcal{P}_{K_0^{-1}}$ satisfies $\alpha \cap (Z_{5\sqrt{d}K_0^{-1/2}} + b_{\alpha^*}) = \phi$, then $c_{\alpha}(B_{K_0^{1/2}}(a)) \leq K_0^{-\frac{d}{2}} c_{\alpha^*}(B_{K_0^{1/2}}(a))$. Thus, for any $x \in B_{K_0^{1/2}}(a)$ we have

$$|Ef(x)| \lesssim \left| \sum_{\alpha \in \mathcal{P}_{K_0^{-1}}: \alpha \cap (Z_{5\sqrt{d}K_0^{-1/2}} + b_{\alpha^*}) \neq \phi} E_{\alpha} f(x) \right| + c_{\alpha^*}(B_{K_0^{1/2}}(a)).$$

By raising to the power p and integrating on the cube $B_{K_0^{1/2}}(a)$, we have

$$\begin{aligned} \|Ef\|_{L^p(B_{K_0^{1/2}}(a))}^p & \lesssim \left\| \sum_{\alpha \in \mathcal{P}_{K_0^{-1}}: \alpha \cap (Z_{5\sqrt{d}K_0^{-1/2}} + b_{\alpha^*}) \neq \phi} E_{\alpha} f(x) \right\|_{L^p(B_{K_0^{1/2}}(a))}^p \\ & \quad + |c_{\alpha^*}(B_{K_0^{1/2}}(a))|^p |B_{K_0^{1/2}}(a)|. \end{aligned}$$

The second term can be easily handled; By Hölder's inequality

$$|c_{\alpha^*}(B_{K_0^{1/2}}(a))|^p |B_{K_0^{1/2}}(a)| \lesssim \int |E_{\alpha^*} f(y)|^p |B_{K_0^{1/2}}(a)|^{\frac{1}{2}} \zeta_{K_0^{1/2}}(y) dy \lesssim \|E_{\alpha^*} f\|_{L^p(w_{B_{K_0^{1/2}}(a)}^2)}^p.$$

If there exists a hyperplane V such that $\Phi(V) = 0$, then we just apply the trivial l^p decoupling to the first term;

$$\left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}} \\ \alpha \cap (Z_{5\sqrt{d}K_0^{-1/2}} + b_{\alpha^*}) \neq \phi}} E_{\alpha} f(x) \right\|_{L^p(B_{K_0^{1/2}}(a))}^p \leq C_p K_0^{\frac{d-1}{2}(p-2)} \sum_{\alpha \in \mathcal{P}_{K_0^{-1}}} \|E_{\alpha} f\|_{L^p(w_{B_{K_0^{1/2}}(a)}^2)}^p.$$

This inequality follows from the fact that the number of cubes α satisfying $\alpha \cap Z_{5\sqrt{d}K_0^{-1/2}} \neq \phi$ is $O(K_0^{\frac{d-1}{2}})$.

This estimate seems to be not sophisticated, but in fact, $K_0^{\frac{d-1}{2}(1-\frac{2}{p})} \lesssim K_0^{\gamma(p)}$ because $S|_V = 0$. We just put $t_i = 1$. Next, suppose that there does not exist such V . In this case, we need the following inequality.

$$(4.3) \quad \|Eg\|_{L^p(B_{K_0^{1/2}}(a))}^p \leq \sum_{i=0}^{d-1} C_{p,\epsilon} K_i^{C\epsilon} A_p(K_i^{t_i})^p \sum_{\beta \in \mathcal{P}_{K_i^{-t_i}}} \|E_{\beta} g\|_{L^p(w_{B_{K_0^{1/2}}(a)}^2)}^p,$$

where $g(\xi) = \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}} \\ \alpha \cap (Z_{5\sqrt{d}K_0^{-1/2} + b_{\alpha^*}) \neq \emptyset}} 1_{\alpha}(\xi)f(\xi)$ and $\frac{1}{4} \leq t_i \leq 1$. Since we are dealing with the second scenario,

$$\begin{aligned} \|Eg\|_{L^p(B_{K_0^{1/2}}(a))}^p &\leq \sum_{i=0}^{d-1} C_{p,\epsilon} K_i^{C\epsilon} A_p(K_i^{t_i})^p \sum_{\beta \in \mathcal{P}_{K_i^{-t_i}}} \|E_{\beta} f\|_{L^p(w_{B_{K_0^{1/2}}(a)}^2)}^p \\ &\quad + C_{p,\epsilon} K_0^{C\epsilon} A_p(K_0)^p \sum_{\beta \in \mathcal{P}_{K_0^{-1}}} \|E_{\beta} f\|_{L^p(w_{B_{K_0^{1/2}}(a)}^2)}^p \end{aligned}$$

If we use the trivial l^p decoupling instead of the inequality (4.3), then we obtain disastrous results. In other words, we have to investigate the decoupling restricted to the set $Z_{5\sqrt{d}K_0^{-1/2}}$ more seriously. This is the key part in this paper, and we will prove the above inequality (4.3) in Section 4.2.

To summarize, in either case, we have

$$\begin{aligned} &\|Ef\|_{L^p(B_{K_0^{1/2}}(a))}^p \\ &\leq \sum_{i=0}^{d-1} C_{p,\epsilon} K_i^{C\epsilon} A_p(K_i^{t_i})^p \sum_{\beta \in \mathcal{P}_{K_i^{-t_i}}} \|E_{\beta} f\|_{L^p(w_{B_{K_0^{1/2}}(a)}^2)}^p \\ &\quad + C_{p,\epsilon} K_0^{C\epsilon} A_p(K_0)^p \sum_{\beta \in \mathcal{P}_{K_0^{-1}}} \|E_{\beta} f\|_{L^p(w_{B_{K_0^{1/2}}(a)}^2)}^p \\ &\quad + C_p K_0^{pd} \iint \sum_{\substack{\alpha, \beta \in \mathcal{P}_{K_0^{-1}} \\ (\alpha, \beta): C_{K_0} \text{-trans}}} \| |E_{\alpha} f(\cdot - y_1) E_{\beta} f(\cdot - y_2)|^{\frac{1}{2}} \|_{L^p(B_{K_0^{\frac{1}{2}}(a)})}^p \zeta_{K_0^{\frac{1}{2}}}(y_1) \zeta_{K_0^{\frac{1}{2}}}(y_2) dy_1 dy_2. \end{aligned}$$

Sum over $B_{K_0^{1/2}} \subset B_N$, and use the definition of $D_{bil}(N, p, \nu)$ and Fubini's theorem. Then we obtain

$$\begin{aligned} \|Ef\|_{L^p(B_N)}^p &\leq \sum_{i=0}^{d-1} C_{p,\epsilon} K_i^{C\epsilon} A_p(K_i^{t_i})^p \sum_{\beta \in \mathcal{P}_{K_i^{-t_i}}} \|E_{\beta} f\|_{L^p(w_{B_N}^2)}^p \\ &\quad + C_{p,\epsilon} K_0^{C\epsilon} A_p(K_0)^p \sum_{\beta \in \mathcal{P}_{K_0^{-1}}} \|E_{\beta} f\|_{L^p(w_{B_N}^2)}^p \\ &\quad + C_p K_0^{2pd} D_{bil}(N, p, C_{K_0})^p \sum_{\theta \in \mathcal{P}_{\delta}} \|E_{\theta} f\|_{L^p(w_{B_N}^2)}^p. \end{aligned}$$

Therefore, this completes the proof of Proposition 4.2. \square

To iterate Proposition 4.2, we need a rescaled version of it. The proof of Proposition 4.3 is similar to that of Proposition 3.2.

Proposition 4.3. Fix $\bar{\epsilon}, \epsilon > 0$ and $p \geq 2$. There exist dyadic numbers K_0, \dots, K_{d-1} such that $\bar{\epsilon} \leq K_{d-1} \leq \dots \leq K_0$ and for any $f : [0, 1]^d \rightarrow \mathbb{C}$, $N \geq K_0$, any dyadic number $t^{-\frac{1}{2}}$ with $\frac{K_0}{N} \leq t \leq 1$ and $\alpha \in \mathcal{P}_t$, we have

$$\begin{aligned} \|E_{\alpha} f\|_{L^p(w_{B_N})}^p &\leq \sum_{i=0}^{d-1} C'_{p,\epsilon} K_i^{C\epsilon} A_p(K_i^{t_i})^p \sum_{\beta \in \mathcal{P}_{tK_i^{-t_i}}} \|E_{\alpha \cap \beta} f\|_{L^p(w_{B_N})}^p \\ &\quad + C'_{p,\epsilon} K_0^{C\epsilon} A_p(K_0)^p \sum_{\beta \in \mathcal{P}_{tK_0^{-1}}} \|E_{\alpha \cap \beta} f\|_{L^p(w_{B_N}^2)}^p \\ &\quad + C'_p K_0^{2pd} D_{bil}(Nt, p, C_{K_0})^p \sum_{\theta \in \mathcal{P}_{\delta}} \|E_{\alpha \cap \theta} f\|_{L^p(w_{B_N})}^p, \end{aligned}$$

where $\frac{1}{4} \leq t_i \leq 1$ and $C'_{p,\epsilon}, C'_p$ are independent of $\bar{\epsilon}$. Here $K_i^{t_i/2}$ are dyadic numbers.

Proof. We define the affine transformation associated with α , which was defined in (3.1). Let C_N be a cylinder of dimensions $t^{1/2}N \times \dots \times t^{1/2}N \times tN \times \dots \times tN$, which was also defined in the proof of Proposition 3.2. Let $g(\xi) = f(L_\alpha^{-1}\xi)t^{\frac{d}{2}-\frac{3d}{2p}}$ as before. We apply Proposition 4.2;

$$\begin{aligned} \|E_\alpha f\|_{L^p(B_N)}^p &= \|Eg\|_{L^p(C_N)}^p \leq \sum_{B_{tN} \cap C_N \neq \emptyset} \|Eg\|_{L^p(B_{tN})}^p \\ &\leq \sum_{i=0}^{d-1} C_{p,\epsilon} K_i^{C_\epsilon} A_p(K_i^{t_i})^p \sum_{\beta \in \mathcal{P}_{K_i^{-t_i}}} \|E_{\beta} g\|_{L^p(\sum_{B_{tN} \subset C_N \neq \emptyset} w_{B_{tN}}^2)}^p \\ &\quad + C_{p,\epsilon} K_0^{C_\epsilon} A_p(K_0)^p \sum_{\beta \in \mathcal{P}_{K_0^{-1}}} \|E_{\beta} f\|_{L^p(\sum_{B_{tN} \subset C_N \neq \emptyset} w_{B_{tN}}^2)}^p \\ &\quad + C_p K_0^{2pd} D_{bil}(tN, p, C_{K_0})^p \sum_{\theta \in \mathcal{P}_{\delta/t}} \|E_{\theta} g\|_{L^p(\sum_{B_{tN} \cap C_N \neq \emptyset} w_{B_{tN}}^2)}^p. \end{aligned}$$

By returning to the original variables, we obtain

$$\begin{aligned} &\leq \sum_{i=0}^{d-1} C'_{p,\epsilon} K_i^{C_\epsilon} A_p(K_i^{t_i})^p \sum_{\beta \in \mathcal{P}_{tK_i^{-t_i}}} \|E_{\alpha \cap \beta} f\|_{L^p(w_{B_N})}^p \\ &\quad + C_{p,\epsilon} K_0^{C_\epsilon} A_p(K_0)^p \sum_{\beta \in \mathcal{P}_{K_0^{-1}}} \|E_{\alpha \cap \beta} f\|_{L^p(w_{B_N})}^p \\ &\quad + C'_p K_0^{2pd} D_{bil}(tN, p, C_{K_0})^p \sum_{\theta \in \mathcal{P}_{\delta}} \|E_{\alpha \cap \theta} f\|_{L^p(w_{B_N})}^p. \end{aligned}$$

Now, Lemma 3.1 completes the proof of Proposition 4.3. \square

Proof of Theorem 4.1. We will apply Proposition 4.3. Fix $\epsilon > 0$. Let $\bar{\epsilon} = 100^d (C'_{p,\epsilon} + C'_p)^{1/\epsilon}$ and $N \geq K_0$. Take $m = \frac{4 \log N}{\log K_{d-1}}$ so that $K_{d-1}^{\frac{m}{8}} = N^{\frac{1}{2}}$. Observe that $|\{\alpha' \in \mathcal{P}_{\delta} : \alpha \subset \alpha'\}| = O(K_0^{10d})$ for any $\alpha \in \mathcal{P}_{\delta K_0^{20}}$. Now, we use Proposition 4.3 repeatedly until the inverse of sidelength of dyadic cubes is in the interval $[N^{-1/2} K_0^5, N^{-1/2} K_0^{10}]$ (Hence, we iterate this at most m times), and then apply the trivial l^p decoupling to change the sidelengths of cubes into $N^{-1/2}$. Then we have

$$\|Ef\|_{L^p(w_{B_N})}^p \lesssim_\epsilon N^\epsilon \sup_{1 < M < N} A_p(M)^p D_{bil}\left(\frac{N}{M}\right)^p \sum_{\Delta \in \mathcal{P}_{\delta}} \|E_{\Delta} f\|_{L^p(w_{B_N})}^p.$$

Thus,

$$D(N, p) \leq C_{\epsilon,p} N^\epsilon A_p(N) \sup_{1 < M < N} \frac{D_{bil}(M, p, C_{K_0})}{A_p(M)}.$$

This completes the proof of Theorem 4.1 \square

4.2. Estimates on the non-transverse set. In this subsection, we always assume that there does not exist a hyperplane V such that $\Phi(V) = 0$. Recall that $Z = \{(\xi_1, \dots, \xi_d) : J((\xi_1, \dots, \xi_d), 0) = 0\}$. The goal of this subsection is to prove the inequality (4.3), namely,

Proposition 4.4. Fix $\bar{\epsilon} > 1$, $\epsilon > 0$ and $p \geq 2$. There exist dyadic numbers K_0, \dots, K_{d-1} such that $K_i \geq \bar{\epsilon}$ and for any $a \in \mathbb{R}^{2d}$, $b_0 \in [0, 1]^d$, for any $f : [0, 1]^d \rightarrow \mathbb{C}$ with $\text{supp}(f) \subset Z_{10\sqrt{d}K_0^{-1/2} + b_0}$,

$$\|Ef\|_{L^p(B_{K_0^{1/2}(a)})}^p \leq \sum_{i=0}^{d-1} C_{p,\epsilon} K_i^{C_\epsilon} A_p(K_i^{t_i})^p \sum_{\beta \in \mathcal{P}_{K_i^{-t_i}}} \|E_{\beta} f\|_{L^p(w_{B_{K_0^{1/2}(a)}}^2)}^p$$

for some $\frac{1}{4} \leq t_i \leq \frac{1}{2}$. Here, $K_i^{t_i/2}$ are dyadic numbers.

A key idea to prove Proposition 4.4 is to approximate the zero set Z by tangent planes. However, this set does not have to be a manifold. Thus, we divide the set Z into two subsets: a manifold part and a singular part. The l^p decoupling associated with the manifold can be dealt with by the above approximation idea. To handle the singular part, we make the singular part into a manifold by deleting a much singular subset. In other words, we divide the singular part into two sets: the manifold part and the much singular part. As before, the l^p decoupling associated with the manifold can be dealt with by the tangent plane approximation argument. We repeat this process $(d-2)$ times more. From the fact that Z is a zero set of a polynomial of degree d , we can deduce that the last singular part is a hyperplane. By using this observation, we can handle the last singular part. This is an outline of the proof of Proposition 4.4.

From Proposition 5.1 and the definition of $\gamma(p)$, we have the following decoupling: for any hyperplane L , for any function $f : \mathbb{R}^{2d-1} \rightarrow \mathbb{C}$ with $\text{supp}(\hat{f}) \subset \mathcal{N}_\delta(S|_L)$, for any $p \geq 2$ and $N \geq 1$

$$\|f\|_{L^p(w_{B_N}^4)} \lesssim N^{\gamma(p)} \left(\sum_{\alpha \in \mathcal{P}_\delta} \|f_{\mathcal{N}_\delta(\alpha)}\|_{L^p(w_{B_N}^4)}^p \right)^{\frac{1}{p}}.$$

Lemma 4.5. Suppose that L is a hyperplane intersecting $[-2, 2]^d$. Then for any $p \geq 2$, dyadic number $K^{1/4} \geq 1$, and for each $g : [0, 1]^d \rightarrow \mathbb{C}$ with $\text{supp}(g) \subset L_{K^{-1/2}}$, we have

$$(4.4) \quad \|Eg\|_{L^p(w_{B_{K^{-1/2}}}^4)} \lesssim A_p(K^{\frac{1}{2}}) \left(\sum_{\alpha \in \mathcal{P}_{K^{-1/2}}} \|E_\alpha g\|_{L^p(w_{B_{K^{-1/2}}}^4)}^p \right)^{\frac{1}{p}}.$$

Proof. By using a change of variables, we can assume that $L = \{\xi : \xi_1 = 0\}$. Next, we can write

$$S = \{(\xi_1, \dots, \xi_d, \tilde{\Phi}(\xi_2, \dots, \xi_d) + \phi(\xi_1, \dots, \xi_d)\xi_1) : \xi \in [0, 1]^d\},$$

where $S|_L = \{(\xi_2, \dots, \xi_d, \tilde{\Phi}(\xi_2, \dots, \xi_d))\}$, $\phi = (\phi_1, \dots, \phi_d)$ and ϕ_i is a polynomial of degree one. Define a function f by

$$\hat{f}(\xi, \Phi(\xi) + (\tau_1, \dots, \tau_d)) = g(\xi) \prod_{i=1}^d 1_{[0, K^{-1/2}/10d]}(\tau_i).$$

Note that

$$f(x) = E_S g(x) \prod_{i=1}^d 1_{[0, \widehat{K^{-1/2}/10d}]}(x_{d+i}).$$

For fixed $x_1 \in [0, K^{1/2}]$, one can see that the support of the Fourier transform of f on x_2, \dots, x_{2d} variables is in the $CK^{-1/2}$ -neighborhood of the surface $S|_L$ for some $C > 0$. Note that $|1_{[0, \widehat{K^{-1/2}/10d}]}(t)| \sim K^{-1/2}$ for $|t| \leq K^{1/2}$. For $x_1 \in [0, K^{1/2}]$

$$\begin{aligned} \|E_S g\|_{L_{x_2, \dots, x_{2d}}^p(B_{K^{1/2}})}^p &\lesssim (K^{1/2})^{dp} \|f\|_{L_{x_2, \dots, x_{2d}}^p(B_{10dK^{1/2}})}^p \\ &\lesssim A_p(K^{\frac{1}{2}}) (K^{1/2})^{dp} \sum_{\alpha \in \mathcal{P}_{K^{-1/2}}} \|f_\alpha\|_{L_{x_2, \dots, x_{2d}}^p(w_{B_{K^{1/2}}}^{10})}^p \\ &\lesssim A_p(K^{\frac{1}{2}}) \sum_{\alpha \in \mathcal{P}_{K^{-1/2}}} \|E_\alpha g\|_{L_{x_2, \dots, x_{2d}}^p(w_{B_{K^{1/2}}}^{10})}^p. \end{aligned}$$

Integrating on $x_1 \in [0, K^{1/2}]$ gives

$$\|E_S g\|_{L^p(B_{K^{1/2}})}^p \lesssim A_p(K^{\frac{1}{2}}) \sum_{\alpha \in \mathcal{P}_{K^{-1/2}}} \|E_\alpha g\|_{L^p(w_{B_{K^{1/2}}}^4)}^p.$$

Now, Lemma 3.1 completes the proof of Lemma 4.5. \square

We need a rescaled version of Lemma 4.5. The proof of Lemma 4.6 is similar to that of Proposition 3.2.

Lemma 4.6. Fix any dyadic number $K^{1/4} \geq 1$. Suppose that $L_{K^{-1}}$ is a rectangular box intersecting $[0, 1]^d$ with dimensions $K^{-1/2} \times \dots \times K^{-1/2} \times K^{-1}$. Then for any $p \geq 2$ and $g : [0, 1]^d \rightarrow \mathbb{C}$ with $\text{supp}(g) \subset L_{K^{-1}}$

$$\|Eg\|_{L^p(w_{B_{K^2}}^2)} \lesssim A_p(K^{\frac{1}{2}}) \left(\sum_{\alpha \in \mathcal{P}_{K^{-3/2}}} \|E_\alpha g\|_{L^p(w_{B_{K^2}}^2)}^p \right)^{\frac{1}{p}}.$$

Proof. Due to translation invariance, we can assume that $B_K = [0, K^2]^{2d}$. By using a translation and a change of variables, we may assume that $L_{K^{-1}}$ is contained in $[-2K^{-1/2}, 2K^{-1/2}]^d$. Define $h(\xi) = K^{-\frac{d}{2} + \frac{3d}{2p}} g(\frac{\xi_1}{K^{1/2}}, \dots, \frac{\xi_d}{K^{1/2}})$. Note that there exists a hyperplane P such that the support of h is contained in the set $P_{K^{-1/2}} \cap [-2, 2]^d$. Let $C_N = [0, K^{\frac{3}{2}}]^d \times [0, K]^d$. By Lemma 4.5,

$$\begin{aligned} \|Eg\|_{L^p(B_{K^2})}^p &= \|Eh\|_{L^p(C_K)}^p = \sum_{B_{K^{1/2}} \subset C_K} \|Eh\|_{L^p(B_{K^{1/2}})}^p \\ &\lesssim A_p(K^{\frac{1}{2}})^p \sum_{\alpha' \in \mathcal{P}_{K^{-1/2}}} \|E_{\alpha'} h\|_{L^p(\sum_{B_{K^{1/2}} \subset C_K} w_{B_{K^{1/2}}}^4)}^p \\ &\lesssim A_p(K^{\frac{1}{2}})^p \left(\sum_{\alpha \in \mathcal{P}_{K^{-3/2}}} \|E_{\alpha} g\|_{L^p(w_{B_{K^2}}^2)}^p \right)^{\frac{1}{p}}. \end{aligned}$$

Now, Lemma 3.1 gives the desired results. \square

Proof of Proposition 4.4. Fix $\bar{\epsilon} > 1$, $\epsilon > 0$ and $p \geq 2$. First, we may assume that $a = 0$ due to translation invariance. Let $K_0 \geq K_1^4 \geq \dots \geq K_{d-1}^{4^{d-1}} \geq \bar{\epsilon}^{4^{d-1}}$, and we fix such K_{d-1} . Other constants will be determined later. Let $F^{(0)}$ be the function F defined at the beginning of Section 4.1. Since F is a nonzero function, one of d functions $\frac{\partial F}{\partial \xi_1}, \dots, \frac{\partial F}{\partial \xi_d}$ is not identically zero. We call this function $F^{(1)}$. Note that the function $F^{(1)}$ is a polynomial of degree $d-1$. Define

$$U^{(1)} = \bigcup_{Q \in \mathcal{P}_{K_1^{-1}} : \exists \xi + b_0 \in 5Q \text{ s.t. } F^{(1)}(\xi) = 0} Q.$$

Since $F^{(1)}$ is a nonzero function, one of d functions $\frac{\partial F^{(1)}}{\partial \xi_1}, \dots, \frac{\partial F^{(1)}}{\partial \xi_d}$ is not identically zero. We call this function $F^{(2)}$, and define a set $U^{(2)}$ as before. By repeating this process, for each $i = 1, \dots, d-1$ we have functions $F^{(i)}$ and sets

$$U^{(i)} = \bigcup_{Q \in \mathcal{P}_{K_i^{-1}} : \exists \xi + b_0 \in 5Q \text{ s.t. } F^{(i)}(\xi) = 0} Q.$$

Note that $U^{(d-1)}$ is contained in the $10\sqrt{d}K_{d-1}^{-1/2}$ -neighborhood of some hyperplane in \mathbb{R}^d because the function $F^{(d-1)}$ is a nonzero polynomial of degree one. We will deal with the decoupling associated with the set $U^{(d-1)}$ by using Lemma 4.5 with $K = K_{d-1}$.

We divide the set $Z_{10\sqrt{d}K_0^{-1/2}} + b_0$ into d sets:

$$Z_{10\sqrt{d}K_0^{-1/2}} + b_0 \subset ((Z_{10\sqrt{d}K_0^{-1/2}} + b_0) \setminus \bigcup_{i=1}^{d-1} U^{(i)}) \cup \left(\bigcup_{i=1}^{d-2} (U^{(i)} \setminus \bigcup_{j=i+1}^{d-1} U^{(j)}) \right) \cup U^{(d-1)}.$$

By the triangle inequality,

$$\left\| \sum_{\alpha \in \mathcal{P}_{K_0^{-1}}} E_{\alpha} f \right\|_{L^p(B_{K_0^{1/2}})}^p \lesssim \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \subset U^{(d-1)}}} E_{\alpha} f \right\|_{L^p(B_{K_0^{1/2}})}^p + \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \cap U^{(d-1)} = \emptyset}} E_{\alpha} f \right\|_{L^p(B_{K_0^{1/2}})}^p.$$

By applying Lemma 4.5 to the first term on the right hand side and using the triangle inequality, we can bound the above term by

$$\begin{aligned}
& \lesssim \sum_{B_{K_{d-1}^{1/2}} \subset B_{K_0^{1/2}}} \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \subset U^{(d-1)}}} E_\alpha f \right\|_{L^p(B_{K_{d-1}^{1/2}})}^p + \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \cap U^{(d-1)} = \phi}} E_\alpha f \right\|_{L^p(B_{K_0^{1/2}})}^p \\
& \lesssim A_p(K_{d-1})^{\frac{p}{2}} \sum_{\beta \in \mathcal{P}_{K_{d-1}^{-1/2}}} \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \subset U^{(d-1)}}} E_{\alpha \cap \beta} f \right\|_{L^p(w_B^2 K_0^{1/2})}^p + \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \cap U^{(d-1)} = \phi}} E_\alpha f \right\|_{L^p(B_{K_0^{1/2}})}^p \\
(4.5) \quad & \lesssim A_p(K_{d-1})^{\frac{p}{2}} \sum_{\beta \in \mathcal{P}_{K_{d-1}^{-1/2}}} \|E_\beta f\|_{L^p(w_B^2 K_0^{1/2})}^p \\
& + A_p(K_{d-1})^{\frac{p}{2}} \sum_{\beta \in \mathcal{P}_{K_{d-1}^{-1/2}}} \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \cap U^{(d-1)} = \phi}} E_{\alpha \cap \beta} f \right\|_{L^p(w_B^2 K_0^{1/2})}^p + \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \cap U^{(d-1)} = \phi}} E_\alpha f \right\|_{L^p(B_{K_0^{1/2}})}^p.
\end{aligned}$$

Put $t_{d-1} = 1/2$. We will not touch the first term any more. We consider the third term. By the triangle inequality,

$$(4.6) \quad \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \cap U^{(d-1)} = \phi}} E_\alpha f \right\|_{L^p(B_{K_0^{1/2}})}^p \lesssim \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \cap (U^{(d-2)} \cup U^{(d-1)}) = \phi}} E_\alpha f \right\|_{L^p(B_{K_0^{1/2}})}^p + \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \subset (U^{(d-2)} \setminus U^{(d-1)})}} E_\alpha f \right\|_{L^p(B_{K_0^{1/2}})}^p.$$

Here, the set $(U^{(d-2)} \setminus U^{(d-1)})$ can be empty. We will prove the following inequality

$$(4.7) \quad \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \subset (U^{(d-2)} \setminus U^{(d-1)})}} E_\theta f \right\|_{L^p(B_{K_0^{1/2}})}^p \leq K_{d-2}^{C_\epsilon} A_p(K_{d-2}^{t_{d-2}})^p \sum_{\gamma \in \mathcal{P}_{K_{d-2}^{-t_{d-2}}}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \subset (U^{(d-2)} \setminus U^{(d-1)})}} E_{\theta \cap \gamma} f \right\|_{L^p(w_B^2 K_0^{1/2})}^p$$

for some t_{d-2} with $\frac{1}{4} \leq t_{d-2} \leq \frac{3}{8}$. For the moment, we assume that the above inequality holds. Note that if a cube $\theta \in \mathcal{P}_{K_0^{-1}}$ doesn't intersect the set $U^{(d-1)}$, then the cube $\gamma \in \mathcal{P}_{K_{d-2}^{-1/4}}$ containing θ also doesn't intersect the set $U^{(d-1)}$ by a property of dyadic cubes. By Hölder's inequality, the above term is bounded by

$$\begin{aligned}
& \leq C_p K_{d-2}^{C_\epsilon} A_p(K_{d-2}^{t_{d-2}})^p \sum_{\gamma \in \mathcal{P}_{K_{d-2}^{-t_{d-2}}}} \|E_\gamma f\|_{L^p(w_B^2 K_0^{1/2})}^p \\
& + C_p K_{d-2}^{C_\epsilon} A_p(K_{d-2}^{t_{d-2}})^p \sum_{\gamma \in \mathcal{P}_{K_{d-2}^{-t_{d-2}}}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \cap (U^{(d-2)} \cup U^{(d-1)}) = \phi}} E_{\theta \cap \gamma} f \right\|_{L^p(w_B^2 K_0^{1/2})}^p.
\end{aligned}$$

Hence, (4.6) is bounded by

$$\begin{aligned}
& \leq C_p K_{d-2}^{C_\epsilon} A_p(K_{d-2}^{t_{d-2}})^p \sum_{\gamma \in \mathcal{P}_{K_{d-2}^{-t_{d-2}}}} \|E_\gamma f\|_{L^p(w_B^2 K_0^{1/2})}^p \\
& + C_p K_{d-2}^{C_\epsilon} A_p(K_{d-2}^{t_{d-2}})^p \sum_{\gamma \in \mathcal{P}_{K_{d-2}^{-t_{d-2}}}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \cap (U^{(d-2)} \cup U^{(d-1)}) = \phi}} E_{\theta \cap \gamma} f \right\|_{L^p(w_B^2 K_0^{1/2})}^p \\
& + C_p \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \cap (U^{(d-2)} \cup U^{(d-1)}) = \phi}} E_\alpha f \right\|_{L^p(B_{K_0^{1/2}})}^p.
\end{aligned}$$

Now, we will obtain (4.7). Let $Z^{(d-2)} = \{b_0 + \xi \in [0, 1]^d : F^{(d-2)}(\xi) = 0\}$. Note that $U^{(d-2)} \subset (Z^{(d-2)})_{100\sqrt{d}K_{d-2}^{-1/2}}$. By using the implicit function theorem, we can take a collection of open cubes $\{V_i\}_{i=1}^{O(1)}$

such that $Z^{(d-2)} \setminus U^{(d-1)} \subset \bigcup_{i=1}^{O(1)} V_i$ and the set $Z^{(d-2)} \cap V_i$ can be represented as a graph of some smooth function. We take K_{d-2} sufficiently large so that

- $(Z^{(d-2)} \setminus U^{(d-1)})_{100\sqrt{d}K_{d-2}^{-1/2}} \subset \bigcup_{i=1}^{O(1)} V_i$
- for any $Q \in \mathcal{P}_{K_{d-2}^{-1/4}}$, there exists V_j such that $(Q \cap Z^{(d-2)} \setminus U^{(d-1)}) \subset V_j$.

Hence, by using Hölder's inequality we may assume that the set $(\text{supp}(f) \cap U^{(d-2)}) \setminus U^{(d-1)} \subset \mathbb{R}^d$ is contained in the $C'K_{d-2}^{-1/2}$ -neighborhood of a smooth hypersurface Z' whose principal curvatures depend on the constant K_{d-1} .

Proposition 4.7. Fix any dyadic number $K^{1/4}$ with $K^{-1} \geq K_{d-2}^{-1/2}$. Let α be a cube of length $K^{-1/2}$ in \mathbb{R}^d intersecting $Z'_{C'K_{d-2}^{-1/2}}$. If $\text{supp}(g) \subset Z'_{C'K_{d-2}^{-1/2}}$, we have

$$\|E_\alpha g\|_{L^p(w_{B_{K^2}}^2)} \leq D_{p,K_{d-1}} A_p(K^{\frac{1}{2}}) \left(\sum_{\beta \in \mathcal{P}_{K^{-3/2}}} \|E_{\beta \cap \alpha} g\|_{L^p(w_{B_{K^2}}^2)}^p \right)^{\frac{1}{p}}.$$

Proof. We may assume that $\alpha \cap Z'_{C'K_{d-1}^{-1/2}} \neq \emptyset$. Take a point $a \in Z'$ so that the distance between a and α is less than $C'K^{-1}$. Draw a tangent plane T of Z' at a in \mathbb{R}^d . Then

$$\alpha \cap Z'_{C'K^{-1}} \subset (a + [-2C'K^{-1/2}, 2C'K^{-1/2}]^d) \cap Z'_{C'K^{-1}},$$

and by Taylor's theorem $\alpha \cap Z'_{C'K^{-1}}$ is contained in $C_{K_{d-1}}K^{-1}$ -neighborhood of T . Moreover, this still holds even if we change the plane T to the truncated plane $T \cap (a + [-2C'K^{-\frac{1}{2}}, 2C'K^{-\frac{1}{2}}]^d)$. Now we apply Lemma 4.6. Then

$$\|E_\alpha g\|_{L^p(w_{B_{K^2}}^2)} \leq D_{p,K_{d-1}} A_p(K^{\frac{1}{2}}) \left(\sum_{\beta \in \mathcal{P}_{K^{-3/2}}} \|E_{\beta \cap \alpha} g\|_{L^p(w_{B_{K^2}}^2)}^p \right)^{\frac{1}{p}}.$$

This completes the proof of Proposition 4.7. \square

Now, we are ready to obtain (4.7). Let K_{d-2} be large enough so that $K_{d-2}^{2\epsilon/2} \frac{100}{\epsilon}$ becomes a dyadic number. Since the number of cubes in $\mathcal{P}_{K_{d-2}^{-4\epsilon}}$ intersecting Z' is $O(K_{d-2}^{2d\epsilon})$, we get

$$\begin{aligned} & \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \subset (U^{(d-2)} \setminus U^{(d-1)})}} E_\theta f \right\|_{L^p(B_{K_0^{1/2}})}^p \lesssim K_{d-2}^{2\epsilon dp} \sum_{\substack{\alpha \in \mathcal{P}_{K_{d-2}^{-4\epsilon}}, \\ \alpha \cap Z'_{K_{d-2}^{-1/2}} \neq \emptyset}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \subset (U^{(d-2)} \setminus U^{(d-1)})}} E_{\theta \cap \alpha} f \right\|_{L^p(B_{K_0^{1/2}})}^p \\ & = K_{d-2}^{2\epsilon dp} \sum_{B_{K_{d-2}^{8\epsilon}} \subset B_{K_0^{1/2}}} \sum_{\substack{\alpha \in \mathcal{P}_{K_{d-2}^{-4\epsilon}}, \\ \alpha \cap Z'_{K_{d-2}^{-1/2}} \neq \emptyset}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \subset (U^{(d-2)} \setminus U^{(d-1)})}} E_{\theta \cap \alpha} f \right\|_{L^p(B_{K_{d-2}^{8\epsilon}})}^p. \end{aligned}$$

By applying Proposition 4.7 with $K = K_{d-2}^{4\epsilon}$ and summing over cubes $B_{K_{d-2}^{8\epsilon}} \subset B_{K_{d-2}^{12\epsilon}}$, we obtain

$$\begin{aligned} & \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \subset (U^{(d-2)} \setminus U^{(d-1)})}} E_\theta f \right\|_{L^p(B_{K_0^{1/2}})}^p \\ & \lesssim_{K_{d-1}} K_{d-2}^{2\epsilon dp} A_p(K_{d-2}^{2\epsilon})^p \sum_{B_{K_{d-2}^{8\epsilon}} \subset B_{K_0^{1/2}}} \sum_{\substack{\beta \in \mathcal{P}_{K_{d-2}^{-6\epsilon}}, \\ \beta \cap Z'_{K_{d-2}^{-\frac{1}{2}}} \neq \emptyset}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \subset (U^{(d-2)} \setminus U^{(d-1)})}} E_{\theta \cap \beta} f \right\|_{L^p(w_{B_{K_{d-2}^{8\epsilon}}}^2)}^p \\ & \lesssim_{K_{d-1}} K_{d-2}^{2\epsilon dp} A_p(K_{d-2}^{2\epsilon})^p \sum_{B_{K_{d-2}^{12\epsilon}} \subset B_{K_0^{1/2}}} \sum_{\substack{\beta \in \mathcal{P}_{K_{d-2}^{-6\epsilon}}, \\ \beta \cap Z'_{K_{d-2}^{-\frac{1}{2}}} \neq \emptyset}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \subset (U^{(d-2)} \setminus U^{(d-1)})}} E_{\theta \cap \beta} f \right\|_{L^p(w_{B_{K_{d-2}^{12\epsilon}}}^2)}^p. \end{aligned}$$

By repeating this process $\lceil \frac{\log(1/3\epsilon) - \log 8}{\log 3/2} \rceil + 1$ times, we obtain

$$\begin{aligned}
& \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \subset (U^{(d-2)} \setminus U^{(d-1)})}} E_{\theta} f \right\|_{L^p(B_{K_0^{\frac{1}{2}}})}^p \\
& \leq C_{p, K_{d-1}, \epsilon} K_{d-2}^{C\epsilon} A_p(K_{d-2}^{2\epsilon(1+\frac{3}{2}+\dots+(\frac{3}{2})^{\lceil \frac{\log(1/3\epsilon) - \log 8}{\log 3/2} \rceil + 1})})^p \sum_{\tau \in \mathcal{P}_{K_{d-2}^{-t_{d-2}}}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \subset (U^{(d-2)} \setminus U^{(d-1)})}} E_{\theta \cap \tau} f \right\|_{L^p(w_B^2 w_{K_0^{\frac{1}{2}}}^{\frac{1}{2}})}^p \\
& \leq C_{p, K_{d-1}, \epsilon} K_{d-2}^{C\epsilon} A_p(K_{d-2}^{t_{d-2}})^p \sum_{\tau \in \mathcal{P}_{K_{d-2}^{-t_{d-2}}}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \subset (U^{(d-2)} \setminus U^{(d-1)})}} E_{\theta \cap \tau} f \right\|_{L^p(w_B^2 w_{K_0^{\frac{1}{2}}}^{\frac{1}{2}})}^p,
\end{aligned}$$

where $t_{d-2} = 6\epsilon(\frac{3}{2})^{\lceil \frac{\log(1/3\epsilon) - \log 8}{\log 3/2} \rceil + 1}$. Let K_{d-2} be large enough so that $C_{p, K_{d-1}, \epsilon} \leq K_{d-2}^{\epsilon}$. This completes the proof of (4.7). Similarly, one can show that

$$\begin{aligned}
& A_p(K_{d-1}^{1/2})^p \sum_{\beta \in \mathcal{P}_{K_{d-1}}^{-1/2}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \cap U^{(d-1)} = \phi}} E_{\theta \cap \beta} f \right\|_{L^p(B_{K_0^{1/2}})}^p \\
& \leq C_p A_p(K_{d-1}^{1/2})^p \sum_{\beta \in \mathcal{P}_{K_{d-1}}^{-1/2}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \cap (U^{(d-2)} \cup U^{(d-1)}) = \phi}} E_{\theta \cap \beta} f \right\|_{L^p(B_{K_0^{1/2}})}^p \\
& \quad + C_p K_{d-2}^{C\epsilon} A_p(K_{d-2}^{t_{d-2}})^p \sum_{\gamma \in \mathcal{P}_{K_{d-2}}^{-t_{d-2}}} \|E_{\gamma} f\|_{L^p(w_B^2 w_{K_0^{1/2}}^{\frac{1}{2}})}^p \\
& \quad + C_p K_{d-2}^{C\epsilon} A_p(K_{d-2}^{t_{d-2}})^p \sum_{\gamma \in \mathcal{P}_{K_{d-2}}^{-t_{d-2}}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \cap (U^{(d-2)} \cup U^{(d-1)}) = \phi}} E_{\theta \cap \gamma} f \right\|_{L^p(w_B^2 w_{K_0^{1/2}}^{\frac{1}{2}})}^p.
\end{aligned}$$

Now, we fix such K_{d-2} .

To summarize,

$$\begin{aligned}
& \|Ef\|_{L^p(B_{K_0^{1/2}})}^p \leq \sum_{i=d-2}^{d-1} C_{p, \epsilon} K_i^{C\epsilon} A_p(K_i^{t_i})^p \sum_{\gamma \in \mathcal{P}_{K_i^{-t_i}}} \|E_{\gamma} f\|_{L^p(w_B^2 w_{K_0^{1/2}}^{\frac{1}{2}})}^p \\
& + C_{p, \epsilon} \sum_{i=d-2}^{d-1} K_i^{C\epsilon} A_p(K_i^{t_i})^p \sum_{\gamma \in \mathcal{P}_{K_i^{-t_i}}} \left\| \sum_{\substack{\theta \in \mathcal{P}_{K_0^{-1}}, \\ \theta \cap (U^{(d-2)} \cup U^{(d-1)}) = \phi}} E_{\theta \cap \gamma} f \right\|_{L^p(w_B^2 w_{K_0^{1/2}}^{\frac{1}{2}})}^p \\
& + C_p \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \cap (U^{(d-2)} \cup U^{(d-1)}) = \phi}} E_{\alpha} f \right\|_{L^p(B_{K_0^{1/2}})}^p.
\end{aligned}$$

Compare this inequality with (4.5). We bound the second and third term as before. By repeating this process, we obtain

$$\begin{aligned} \|Ef\|_{L^p(B_{K_0^{1/2}})}^p &\leq \sum_{i=0}^{d-1} C_{p,\epsilon} K_i^{C\epsilon} A_p(K_i^{t_i})^p \sum_{\gamma \in \mathcal{P}_{K_i^{-t_i}}} \|E_\gamma f\|_{L^p(w_{K_0^{1/2}}^2)}^p \\ &+ \sum_{i=0}^{d-1} C_{p,\epsilon} K_i^{C\epsilon} A_p(K_i^{t_i})^p \sum_{\gamma \in \mathcal{P}_{K_i^{-t_i}}} \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \cap (\bigcup_{l=1}^{d-1} U^{(l)} \cup (Z_{10\sqrt{d}K_0^{-1/2} + b_0)) = \phi}} E_{\alpha \cap \gamma} f \right\|_{L^p(B_{K_0^{1/2}})}^p \\ &+ C_p \left\| \sum_{\substack{\alpha \in \mathcal{P}_{K_0^{-1}}, \\ \alpha \cap (\bigcup_{l=1}^{d-1} U^{(l)} \cup (Z_{10\sqrt{d}K_0^{-1/2} + b_0)) = \phi}} E_{\alpha \cap \gamma} f \right\|_{L^p(B_{K_0^{1/2}})}^p. \end{aligned}$$

Note that the second term and third term are zero because of the support condition of f . Therefore, this completes the proof of Proposition 4.4. \square

5. THE EQUIVALENT FORMULATIONS

Fix positive integers d, m . Let $S = \{(\xi, \Phi_1(\xi), \dots, \Phi_m(\xi)) : \xi \in [0, 1]^d\}$ be a d -dimensional surface in \mathbb{R}^{d+m} , and let Φ_i be homogeneous polynomials of degree two, possibly zero. For $\delta > 0$ and $Q \subset [0, 1]^d$ we define the δ -neighborhood of S above Q to be

$$\mathcal{N}_\delta(Q) = \{(\xi, \Phi_1(\xi) + t_1, \dots, \Phi_m(\xi) + t_m) : \xi \in Q, -\delta \leq t_1, \dots, t_m \leq \delta\}.$$

For a function f and a measurable set $E \subset \mathbb{R}^{2d}$, we denote by $f_E = (\hat{f}1_E)^\vee$ the Fourier restriction to the set E . Here, the notation \vee is the Fourier inverse transform. For $x \in \mathbb{R}^{d+m}$, we use the notation $x = (x', x'')$, where $x' = (x_1, \dots, x_d)$ and $x'' = (x_{d+1}, \dots, x_{d+m})$.

Take a collection of non-negative smooth functions $\{\chi(\cdot + k)\}_{k \in \mathbb{N}}$ such that $\chi(\xi) = 1$ if $\xi \in [-1, 1]$ and $\chi(\xi) = 0$ if $\xi \in [-2, 2]^c$. For each cube $Q = (i_1, \dots, i_d) + [0, \delta^{1/2}]^d \in \mathcal{P}_\delta$, we define a function Ξ_Q to be

$$\widehat{\Xi_Q}(\xi_1, \dots, \xi_{d+m}) = \prod_{k=1}^d \frac{\chi(N^{1/2}(\xi_k - i_k))}{\sum_{m \in \mathbb{Z}^d} \chi(N^{1/2}\xi_k - m)} \prod_{k=1}^m \chi(N(\xi_{k+d} - \Phi_k(\xi_1, \dots, \xi_d))).$$

Note that $\|\Xi_Q\|_1 \sim 1$, $\text{supp}(\widehat{\Xi_Q}) \subset \mathcal{N}_{5\delta}(5Q)$ and $\sum_{Q \in \mathcal{P}_\delta} \widehat{\Xi_Q}(\xi) = 1$ for all $\xi \in \mathcal{N}_\delta([0, 1]^d)$.

Fix $\nu > 0$. For any $2 \leq p < \infty$, $1 \leq l < \infty$ and any dyadic number $N^{1/2} \geq 1$, we denote by $\tilde{D}(N, p, l)$ the smallest constant such that the following decoupling holds;

$$\|f\|_{L^p(w_{B_N}^l)} \leq \tilde{D}(N, p, l) \left(\sum_{Q \in \mathcal{P}_\delta} \|f_{\mathcal{N}_\delta(Q)}\|_{L^p(w_{B_N}^l)}^p \right)^{\frac{1}{p}}$$

for each $f : \mathbb{R}^{d+m} \rightarrow \mathbb{C}$ with Fourier support in $\mathcal{N}_\delta([0, 1]^d)$. Similarly, we denote by $\tilde{D}_{bil}(N, p, \nu)$ the smallest constant such that the following decoupling holds;

$$\|f_1 f_2\|^{\frac{1}{2}}_{L^p(w_{B_N}^2)} \leq \tilde{D}_{bil}(N, p, \nu) \prod_{i=1}^2 \left(\sum_{Q \in \mathcal{P}_\delta} \|(f_i)_{\mathcal{N}_\delta(Q)}\|_{L^p(w_{B_N}^2)}^p \right)^{\frac{1}{2p}}$$

for any $f_i : \mathbb{R}^{d+m} \rightarrow \mathbb{C}$ with Fourier support in $\mathcal{N}_\delta(Q_i)$, where Q_1, Q_2 are any ν -transverse dyadic cubes in $[0, 1]^d$.

We denote by $D^{(1)}(N, p, l)$ the smallest constant such that the following decoupling holds;

$$\|f\|_{L^p(w_{B_N}^l)} \leq D^{(1)}(N, p, l) \left(\sum_{Q \in \mathcal{P}_\delta} \|f * \Xi_Q\|_{L^p(w_{B_N}^l)}^p \right)^{\frac{1}{p}}$$

for any $f : \mathbb{R}^{d+m} \rightarrow \mathbb{C}$ with Fourier support in $\mathcal{N}_\delta([0, 1]^d)$, and we denote by $D^{(2)}(N, p)$ the smallest constant such that the following decoupling holds;

$$\|f\|_{L^p(\mathbb{R}^{2d})} \leq D^{(2)}(N, p) \left(\sum_{Q \in \mathcal{P}_\delta} \|f * \Xi_Q\|_{L^p(\mathbb{R}^{2d})}^p \right)^{\frac{1}{p}}$$

for any $f : \mathbb{R}^{d+m} \rightarrow \mathbb{C}$ with Fourier support in $\mathcal{N}_\delta([0, 1]^d)$. For each rectangular box R , we denote by a_R an affine map taking $[0, 1]^{d+m}$ to the rectangle R . Let η be a Schwartz function such that $\eta \sim 1$ on $B(0, 1)$ and Fourier support is in $B(0, c)$ for some $c > 1$, and let $\eta_R = \eta \circ a_R^{-1}$. By using the observation that the Fourier support of $(f * \Xi_Q)\eta_{B_N}$ is in $\mathcal{N}_{10\delta}(10Q)$, one can see that $D^{(1)}(N, p, l) \lesssim_l D^{(2)}(N, p) \lesssim_l D^{(1)}(N, p, l)$. Hence, we have $D^{(1)}(N, p, l) \sim_l D^{(1)}(N, p, 1)$.

The proof of Proposition 5.1 is identical to that of Theorem 5.1 in [8].

Proposition 5.1. Let S be a d -dimensional surface in \mathbb{R}^{d+m} with $\|\Phi\|_{L^\infty([0, 1]^d)} \leq 1$. Then for any $\nu > 0$, $l \geq 1$, $N \geq 1$ and $p \geq 2$

$$C_{p,l}^{-1} D(N, p, l) \leq \tilde{D}(N, p, l) \leq C_{p,l} D(N, p, l), \quad D_{bil}(N, p, \nu) \leq C_{p,\nu} \tilde{D}_{bil}(N, p, \nu).$$

Proof. We may assume that the cube B_N in the definition of $D(N, p, l)$ is $[0, N]^{d+m}$. Define a function f to be

$$\hat{f}(\xi, \Phi(\xi) + (\tau_1, \dots, \tau_m)) = g(\xi) \prod_{i=1}^m 1_{[0, \delta/10]}(\tau_i)$$

Note that

$$f(x_1, \dots, x_{d+m}) = E g(x_1, \dots, x_{d+m}) \prod_{i=1}^m \int_0^{\delta/10} e(tx_{i+d}) dt,$$

and

$$f_{\mathcal{N}_\delta(Q)}(x) = E_Q g(x) \prod_{i=1}^m \int_0^{\delta/10} e(tx_{i+d}) dt.$$

Note also that $|\int_0^{\delta/10} e(tx_{i+d}) dt| \sim \delta$ if $x_{i+d} \in [0, N]$. These give

$$\begin{aligned} \|Eg\|_{L^p(B_N)} &\lesssim \delta^{-d} \|f\|_{L^p(w_{B_N}^l)} \lesssim \delta^{-d} \tilde{D}(N, p, l) \left(\sum_{Q \in \mathcal{P}_\delta} \|f_{\mathcal{N}_\delta(Q)}\|_{L^p(w_{B_N}^l)}^p \right)^{\frac{1}{p}} \\ &\lesssim \tilde{D}(N, p, l) \left(\sum_{Q \in \mathcal{P}_\delta} \|E_Q g\|_{L^p(w_{B_N}^l)}^p \right)^{\frac{1}{p}}. \end{aligned}$$

Now, Lemma 3.1 gives $D(N, p, l) \lesssim \tilde{D}(N, p, l)$. Similarly, one can get $D_{bil}(N, p, \nu) \lesssim \tilde{D}_{bil}(N, p, \nu)$.

Now, we will show that $\tilde{D}(N, p, l) \lesssim D(N, p, l)$. By a change of variables,

$$\begin{aligned} f(x_1, \dots, x_{d+m}) &= \int_{\mathcal{N}_\delta([0, 1]^d)} \hat{f}(\xi, \tau) e((\xi, \tau) \cdot x) d\xi d\tau \\ &= \sum_{Q \in \mathcal{P}_\delta} \int_{Q \times [-\delta, \delta]^m} \hat{f}(\xi, \Phi(\xi) + \tau) e((\xi, \Phi(\xi)) \cdot x) e(\tau \cdot x'') d\xi d\tau. \end{aligned}$$

We will deal with the term $e(\tau \cdot (x_{d+1}, \dots, x_{d+m}))$ by using the Taylor expansion

$$e(\tau \cdot (x_{d+1}, \dots, x_{d+m})) = \prod_{i=1}^m \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{2ix_{i+d}}{N} \right)^j \left(\frac{N\tau_i}{2} \right)^j.$$

By putting this, for $x \in B_N$ we have

$$|f(x)| \leq \sum_{j_1, \dots, j_m} \frac{2^{j_1} \dots 2^{j_m}}{j_1! \dots j_m!} \left| \sum_{Q \in \mathcal{P}_\delta} E_Q g_{j_1, \dots, j_m}(x) \right|,$$

where

$$g_{j_1, \dots, j_m}(\xi) = \int_{[-\delta, \delta]^m} \hat{f}(\xi, \Phi(\xi) + \tau) \prod_{i=1}^m \left(\frac{N\tau_i}{2} \right)^{j_i} d\tau_1 \dots d\tau_m.$$

From the definition of $D(N, p, l)$, we have

$$\|f\|_{L^p(B_N)} \lesssim D(N, p, l) \sum_{j_1, \dots, j_m} \frac{2^{j_1} \dots 2^{j_m}}{j_1! \dots j_m!} \left(\sum_{Q \in \mathcal{P}_\delta} \|E_Q g_{j_1, \dots, j_m}\|_{L^p(w_{B_N}^l)}^p \right)^{1/p}.$$

Fix $Q = c + [0, \delta^{1/2}] = (c_1, \dots, c_d) + [0, \delta^{1/2}]^d$. By Lemma 3.1, the inequality

$$\|E_Q g_{j_1, \dots, j_m}\|_{L^p(w_{B_N}^l)} \lesssim \|f_{\mathcal{N}_\delta(Q)}\|_{L^p(w_{B_N}^l)},$$

which is uniform over j_1, \dots, j_m , implies the desired results, and this follows from

$$(5.1) \quad \|E_Q g_{j_1, \dots, j_m}\|_{L^p(B_N)} \lesssim \|f_{\mathcal{N}_\delta(Q)}\|_{L^p(w_{B_N}^l)}.$$

We take a Schwartz function $M_j(t)$ which agrees with t^j on $[-1/2, 1/2]$ and satisfies the derivative bound

$$\left\| \frac{d^s}{dt^s} M_j \right\|_{L^\infty(\mathbb{R})} \lesssim_s 1,$$

uniformly over $j \geq 1$, for each $s \geq 0$. The following equality gives the relation between $E_Q g_{j_1, \dots, j_m}$ and $f_{\mathcal{N}_\delta(Q)}$

$$(5.2) \quad \begin{aligned} E_Q g_{j_1, \dots, j_m}(x) &= \int_{\mathcal{N}_\delta(Q)} \hat{f}(\xi, \tau) m_{j_1, \dots, j_m}(\xi, \tau) e((\xi, \tau) \cdot x) d\xi d\tau \\ &= (f_{\mathcal{N}_\delta(Q)} * (m_{j_1, \dots, j_m})^\vee)(x). \end{aligned}$$

Here, m_{j_1, \dots, j_m} is defined by

$$m_{j_1, \dots, j_m}(\xi, \tau) = e((\Phi(\xi) - \tau) \cdot (x_{d+1}, \dots, x_{d+m})) \prod_{i=1}^m M_{j_i} \left(\frac{N(\tau_i - \Phi_i(\xi))}{2} \right) \prod_{i=1}^d H \left(\frac{\xi_i - c_i}{\delta^{1/2}} \right),$$

and $H(\xi_i)$ is a Schwartz function equal to 1 on $[0, 1]$ and 0 on $(-\infty, -1] \cup [2, \infty)$. The above equality immediately follows from a change of variables. Now, we will estimate $(m_{j_1, \dots, j_m})^\vee(y)$. A change of variables gives

$$|(m_{j_1, \dots, j_m})^\vee(y)| = \left| \prod_{i=1}^m \frac{2}{N} \widehat{M}_{j_i} \left(\frac{2(x_{d+i} - y_{d+i})}{N} \right) \right| \int e((\xi, \Phi(\xi + c)) \cdot y) \prod_{i=1}^d H \left(\frac{\xi_i}{\delta^{1/2}} \right) d\xi.$$

Since Φ_i are quadratic polynomials, we can write $\Phi(\xi + c) = \Phi(\xi) + \Phi(c) + \xi A$ for some $d \times m$ matrix A (with transpose A^T). Hence, we can write the second term as

$$\frac{1}{N^{d/2}} \left| \int e((\xi, \Phi(\xi)) \cdot \left(\frac{y' + A^T y''}{N^{1/2}}, \frac{y''}{N} \right)) \prod_{i=1}^d H(\xi_i) d\xi \right|.$$

Thus, by integration by parts and the construction of the function M_j , for $x \in B_N$

$$|(m_{j_1, \dots, j_d})^\vee(y)| \lesssim \frac{\delta^m}{1 + \left| \frac{y'' - x''}{N} \right|^{500(d+m)l}} \frac{\delta^{d/2}}{1 + \left| \frac{y' + A^T y''}{N^{1/2}} \right|^{500(d+m)l}}.$$

Now, we are ready to obtain (5.1). By (5.2), Young's inequality and the above inequality

$$\|E_Q g_{j_1, \dots, j_d}\|_{L^p(B_N)}^p \lesssim \|f_{\mathcal{N}_\delta(Q)} * (m_{j_1, \dots, j_m})^\vee\|_{L^p(B_N)}^p \lesssim \|f_{\mathcal{N}_\delta(Q)}\|_{L^p(w_{B_N}^l)}^p.$$

This completes the proof of Proposition 5.1. \square

By using the fact that $\{\widehat{\Xi}_Q\}$ forms a partition of unity on $\mathcal{N}_\delta([0, 1]^d)$ and has a finitely overlapping property, one can prove the following proposition.

Proposition 5.2. Let S be a d -dimensional surface in \mathbb{R}^{d+m} with $\|\Phi\|_{L^\infty([0, 1]^d)} \leq 1$. Then for any $\nu > 0$, $l \geq 1$, $N \geq 1$ and $p \geq 2$

$$C_{p,l}^{-1} \tilde{D}(N, p, l) \leq D^{(1)}(N, p, l) \leq C_{p,l} \tilde{D}(N, p, l), \quad \tilde{D}_{bil}(N, p, \nu) \leq C_{p,\nu} D_{bil}^{(1)}(N, p, \nu).$$

By using Proposition 5.1 and 5.2, we can see $D(N, p, 1) \sim_l D(N, p, l)$. The proof of Proposition 5.3 is identical to that of Proposition 3.2. Hence, we will omit the detail.

Proposition 5.3 (Parabolic rescaling). Suppose that two numbers δ, σ with $0 < \delta \leq \sigma$ are dyadic number, and let $\tau = a + [0, \sigma^{1/2}]^d \in \mathcal{P}_\sigma$. Then for each $f : \mathbb{R}^{d+m} \rightarrow \mathbb{C}$ with Fourier support in $\mathcal{N}_\sigma(\tau)$, we have

$$\|f\|_{L^p(\mathbb{R}^{d+m})} \lesssim D\left(\frac{\sigma}{\delta}, p\right) \left(\sum_{\theta \in \mathcal{P}_\delta, \theta \subset \tau} \|f * \Xi_\theta\|_{L^p(\mathbb{R}^{d+m})}^p \right)^{\frac{1}{p}}.$$

6. THE WAVE PACKET DECOMPOSITION

In this section, we will obtain the wave packet decomposition, which will be used to prove Proposition 7.3. The proof of the wave packet decomposition is well known, and we will follow the proof in [13] and [14].

For each rectangle R , we denote by a_R an affine map taking $[0, 1]^{2d}$ to the rectangle R . We take a Schwartz function ϕ such that the function is strictly positive in $B_2(0)$, the Fourier support is in $B_{C_0}(0)$ and $\sum_{n \in \mathbb{Z}^{2d}} \phi(\cdot + n)^2 = 1$ for some $C_0 > 0$. Let $\phi_R = \phi \circ a_R^{-1}$.

Let S be a d -dimensional nondegenerate surface in \mathbb{R}^{2d} . Fix $\nu, \delta > 0$. We say that two sets $E, F \subset \mathcal{N}_\delta([0, 1]^d)$ are ν -transverse if $\pi(E)$ and $\pi(F)$ are ν -transverse.

Definition 6.1. Let $\theta = c + [0, \delta^{1/2}]^d$, $c \in \mathbb{R}^d$. We take a rectangular box R_θ of dimensions $C\delta^{1/2} \times \dots \times C\delta^{1/2} \times C\delta \times \dots \times C\delta$ such that

- (1) $\mathcal{N}_\delta(\theta) \subset R_\theta$
- (2) the short directions are parallel to the subspace spanned by $m_1(c), \dots, m_d(c)$

for some constant C independent of δ and the choice of θ . We denote the dual set of R_θ by R_θ^* , and we write $R_\theta^* \parallel \theta$ if the above conditions are satisfied.

Lemma 6.2 (The wave packet decomposition). Fix $\epsilon > 0$. Let $N \geq 1$, and let Q be a cube with a side length of $2N$ in \mathbb{R}^{2d} . Let f be a function with $\text{supp } \hat{f} \subset \mathcal{N}_{N^{-1}}([0, 1]^d)$. Assume that

$$\sup_{\theta \in \mathcal{P}_{N^{-1}}} \|f * \Xi_\theta\|_{L^\infty(\mathbb{R}^{2d})} \leq A.$$

Then we can decompose f into

$$(6.1) \quad f(x) = \sum_{AN^{-10d} \lesssim 2^m \lesssim A} \sum_{j=1}^{O(\log N)} f^{[j, m]}(x) + g(x), \quad x \in \mathbb{R}^{2d},$$

such that $f^{[j, m]}$ and g satisfy the following:

- (1) The function g is an essentially error function. More precisely,

$$\|g\|_{L^\infty(Q)} \lesssim_\epsilon N^{-8d} A.$$

- (2) For every $2 \leq p < \infty$, j and m , we have

$$(6.2) \quad \|f^{[j, m]}\|_{L^2(\mathbb{R}^{2d})}^2 \left(\sum_{\theta \in \mathcal{P}_{N^{-1}}} \|f^{[j, m]} * \Xi_\theta\|_{L^\infty(\mathbb{R}^{2d})}^2 \right)^{\frac{p-2}{2}} \lesssim \left(\sum_{\theta \in \mathcal{P}_{N^{-1}}} \|f * \Xi_\theta\|_{L^p(\mathbb{R}^{2d})}^2 \right)^{\frac{p}{2}}.$$

By using the reverse inequality (6.2), we can recover the original function f from the packets.

Proof. We decompose f by dividing a frequency space; $f = \sum_{\theta \in \mathcal{P}_{N^{-1}}} f * \Xi_\theta$. Next, we decompose each $f * \Xi_\theta$ by splitting a physical space; $f * \Xi_\theta = \sum_{\pi \in L: \pi \parallel \theta} (f * \Xi_\theta) \phi_\pi^2$, where $L = \{\pi\}$ is a tiling of \mathbb{R}^{2d} . We define $\mathcal{L}_{\theta, Q} = \{\pi \in L : \pi \parallel \theta, \pi \cap 2N^\epsilon Q \neq \emptyset\}$. Note that $|\mathcal{L}_{\theta, Q}| \lesssim N^d$.

Now, we exclude error terms. Define

$$g = \sum_{\theta \in \mathcal{P}_{N^{-1}}} \left(\sum_{\pi \in L: \pi \not\parallel \theta, Q} (f * \Xi_\theta) \phi_\pi^2 + \sum_{\substack{\pi \in \mathcal{L}_{\theta, Q} \\ \|(f * \Xi_\theta) \phi_\pi\|_{L^\infty(\mathbb{R}^{2d})} \leq AN^{-10d}}} (f * \Xi_\theta) \phi_\pi^2 \right).$$

To show the first property in Lemma 6.2, we use a Schwartz tail of the function ϕ_π ;

$$\begin{aligned} & \|g\|_{L^\infty(Q)} \\ & \leq \sum_{\theta} \left(\sum_{\substack{\pi \in L: \\ \pi \notin \mathcal{L}_{\theta,Q}}} \|(f * \Xi_\theta)\phi_\pi^2\|_{L^\infty(Q)} + \sum_{\substack{\pi \in \mathcal{L}_{\theta,Q} \\ \|(f * \Xi_\theta)\phi_\pi\|_{L^\infty(\mathbb{R}^{2d})} \leq AN^{-10d}}} \|(f * \Xi_\theta)\phi_\pi^2\|_{L^\infty(Q)} \right) \\ & \lesssim_\epsilon N^{-100d} \sup_{\theta} \|f * \Xi_\theta\|_{L^\infty(\mathbb{R}^{2d})} + N^{2d} AN^{-10d} \lesssim N^{-8d} A. \end{aligned}$$

Hence, the first property follows.

The main term can be written as

$$f - g = \sum_{\theta \in \mathcal{P}_{N^{-1}}} \sum_{\substack{\pi \in \mathcal{L}_{\theta,Q} \\ \|(f * \Xi_\theta)\phi_\pi\|_{L^\infty(\mathbb{R}^{2d})} > AN^{-10d}}} (f * \Xi_\theta)\phi_\pi^2.$$

For each cube $\theta \in \mathcal{P}_{N^{-1}}$ and $m \in \mathbb{Z}$, we define

$$\mathcal{L}_{\theta,Q}^m = \{\pi \in \mathcal{L}_{\theta,Q} : 2^m < \|(f * \Xi_\theta)\phi_\pi\|_{L^\infty(\mathbb{R}^{2d})} \leq 2^{m+1}\}.$$

Since $\|(f * \Xi_\theta)\phi_\pi\|_{L^\infty(\mathbb{R}^{2d})} \leq \|f * \Xi_\theta\|_{L^\infty(\mathbb{R}^{2d})} \leq A$, we can see $2^m \leq A$ if the set $\mathcal{L}_{\theta,Q}^m$ is non-empty. Next, for each $j \in \mathbb{Z}$ we define

$$E_{j,m} = \{\theta \in \mathcal{P}_{N^{-1}} : 2^j < |\mathcal{L}_{\theta,Q}^m| \leq 2^{j+1}\}.$$

We can also see that $2^j \leq CN^d$ for some $C > 0$ if the $E_{j,m}$ is non-empty, so the set $E_{j,m}$ is non-empty only for $j \lesssim \log N$. Now, we define the functions associated with (j, m) by

$$f^{[j,m]} = \sum_{\theta \in E_{j,m}} \sum_{\pi \in \mathcal{L}_{\theta,Q}^m} (f * \Xi_\theta)\phi_\pi^2.$$

We write $\mathcal{L}^{j,m} = \cup_{\theta \in E_{j,m}} \mathcal{L}_{\theta,Q}^m$. Note that the equality (6.1) holds and $|\mathcal{L}^{j,m}| = |E_{j,m}| |\mathcal{L}_{\theta,Q}^m|$.

We will show the inequality (6.2). Observe that

$$\|f^{[j,m]}\|_{L^2(\mathbb{R}^{2d})}^2 \lesssim 2^{2m} N^{3d/2} |\mathcal{L}^{j,m}| \quad \text{and} \quad \sum_{\theta} \|f^{[j,m]} * \Xi_\theta\|_{L^\infty(\mathbb{R}^{2d})}^2 \lesssim 2^{2m} |E_{j,m}|.$$

These inequalities follow from an orthogonality property. The second property in Lemma 6.2 follows from

$$2^{mp} N^{\frac{3d}{2}} |\mathcal{L}^{j,m}| |E_{j,m}|^{\frac{p-2}{2}} \lesssim \left(\sum_{\theta \in \mathcal{P}_{N^{-1}}} \|f * \Xi_\theta\|_{L^p(\mathbb{R}^{2d})}^2 \right)^{\frac{p}{2}}.$$

Note that $|\mathcal{L}_{\theta_1,Q}^m| \sim |\mathcal{L}_{\theta_2,Q}^m| \sim 2^j$ for any θ_1 and θ_2 in $E_{j,m}$ and $|\mathcal{L}^{j,m}| \lesssim 2^j |E_{j,m}|$. This implies that for any $\theta \in E_{j,m}$

$$\begin{aligned} 2^{mp} N^{\frac{3d}{2}} |\mathcal{L}^{j,m}| |E_{j,m}|^{\frac{p-2}{2}} & \lesssim 2^{mp} N^{\frac{3d}{2}} 2^j |E_{j,m}|^{\frac{p}{2}} \lesssim |E_{j,m}|^{\frac{p}{2}} \sum_{\pi \in \mathcal{L}_{\theta,Q}^m} N^{\frac{3d}{2}} 2^{mp} \\ & \lesssim |E_{j,m}|^{\frac{p}{2}} \sum_{\pi \in \mathcal{L}_{\theta,Q}^m} \|(f * \Xi_\theta)\phi_\pi\|_{L^\infty(\mathbb{R}^{2d})}^p N^{\frac{3d}{2}} \\ & \lesssim |E_{j,m}|^{\frac{p}{2}} \sum_{\pi \in \mathcal{L}_{\theta,Q}^m} \|(f * \Xi_\theta)\phi_\pi\|_{L^p(\mathbb{R}^{2d})}^p \lesssim |E_{j,m}|^{\frac{p}{2}} \|f * \Xi_\theta\|_{L^p(\mathbb{R}^{2d})}^p \end{aligned}$$

by Bernstein's inequality and $\sum_{\pi} |\phi_\pi|^p \lesssim 1$. Raising to the power $\frac{2}{p}$ and summing over all $\theta \in E_{j,m}$ lead to the desired inequality. \square

7. PROOF OF THEOREM 1.1

We will follow Guth's multiscale argument, which was given in [16]. The only difference between our proof and the proof in [16] is that we obtain the l^p decoupling instead of the l^2 decoupling, but this will not make trouble. Let S be a d -dimensional nondegenerate surface in \mathbb{R}^{2d} .

For simplicity, we write

$$\|f\|_{L^{p,N}(\mathbb{R}^{2d})} = \left(\sum_{\theta \in \mathcal{P}_\delta} \|f * \Xi_\theta\|_{L^p(\mathbb{R}^{2d})}^p \right)^{\frac{1}{p}}, \quad \|f\|_{L^{p,N}(w_{B_N}^2)} = \left(\sum_{\theta \in \mathcal{P}_\delta} \|f * \Xi_\theta\|_{L^p(w_{B_N}^2)}^p \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$.

Theorem 7.1. Suppose that the inequality (4.1) holds for some p_0 ;

$$\gamma_{lin}(p_0) \leq \gamma_{bil}(p_0).$$

Then, we have

$$\gamma_{bil}(p_0) \leq \max\left(\frac{d}{2} - \frac{2d}{p_0}, \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p_0}\right)\right).$$

Since Theorem 7.1 implies Theorem 1.1, it suffices to prove Theorem 7.1.

Proposition 7.2. Fix $N \geq 1$ and $\nu > 0$. If $\text{supp } \hat{f}_i \subset \mathcal{N}_{N-1}([0,1]^d)$ and the Fourier supports of f_i are ν -transverse, then for each $p \geq 2$

$$\int_{B_N} \prod_{i=1}^2 \|f_i\|_{L^2(B_{N^{1/2}}(x))}^{\frac{p}{2}} dx \lesssim_\nu \max(N^{d(2-\frac{p}{2})}, N^{d(1-\frac{p}{4})}) \prod_{i=1}^2 \|f_i\|_{L^2(\mathbb{R}^{2d})}^{\frac{p}{2}}.$$

The proof of Proposition 7.2 is very similar to that of Proposition 2.1 in [2], that of Proposition 4.7 in [1] and that of Lemma 4.4 in [19].

Proof. Note that the two constants $N^{d(2-\frac{p}{2})}, N^{d(1-\frac{p}{4})}$ are equal when $p = 4$. Moreover, if $p = 2$, then the above inequality immediately follows from Cauchy-Schwartz's inequality.

Suppose that $p \geq 4$. Let η be the function defined at the beginning of Section 5. Since $\{(\widehat{f_i * \Xi_\theta}) * \eta_{B_{N^{1/2}}(x)}\}_{\theta \in \mathcal{P}_{N-1}}$ is a finitely overlapping collection, we have

$$\int_{B_N} \prod_{i=1}^2 \|f_i\|_{L^2(B_{N^{1/2}}(x))}^{\frac{p}{2}} dx \lesssim \int_{B_N} \prod_{i=1}^2 \left(\sum_{\theta \in \mathcal{P}_{N-1}} \|f_i * \Xi_\theta\|_{L^2(\eta_{B_{N^{1/2}}(x)})}^2 \right)^{\frac{p}{4}} dx.$$

Hence, it suffices to prove that

$$\int_{B_N} \prod_{i=1}^2 \left(\sum_{\theta \in \mathcal{P}_{N-1}} \|f_i * \Xi_\theta\|_{L^2(\eta_{B_{N^{1/2}}(x)})}^2 \right)^{\frac{p}{4}} dx \lesssim_\nu N^{d(1-\frac{p}{4})} \prod_{i=1}^2 \|f_i\|_{L^2(\mathbb{R}^{2d})}^{\frac{p}{2}}.$$

Moreover, by a Schwartz tail of η , it suffices to show that for any $c_1, c_2 \in \mathbb{R}^{2d}$

$$\int_{B_N} \prod_{i=1}^2 \left(\sum_{\theta \in \mathcal{P}_{N-1}} \|f_i * \Xi_\theta\|_{L^2(B_{cN^{1/2}}(x+c_i))}^2 \right)^{\frac{p}{4}} dx \lesssim_\nu N^{d(1-\frac{p}{4})} \prod_{i=1}^2 \|f_i\|_{L^2(\mathbb{R}^{2d})}^{\frac{p}{2}}$$

for some sufficiently small $c > 0$. For each $\theta \in \mathcal{P}_{N-1}$, we take a rectangular box θ_0^* so that $\theta_0^* \parallel \theta$ and $\mathcal{N}_{5\delta}(5\theta) \subset \theta_0^*$. Note that $\eta_{\theta_0} = 1$ on θ_0 and

$$|\eta_{\theta_0}^\vee(x+y)| \lesssim N^{-3d/2} \chi_{\theta_0^*}(-x)$$

for all x, y with $y \in [0, cN^{1/2}]^{2d}$. Define $(\tilde{f}_{i,\theta})^\vee(\xi) = e(-\xi \cdot c_i) (\widehat{f_i * \Xi_\theta})(\xi) / \eta_{\theta_0}(\xi)$. By Cauchy-Schwartz's inequality and the above inequality, we have

$$\begin{aligned} |(f_i * \Xi_\theta)(x+y)|^2 &\lesssim \left(\int_{\mathbb{R}^{2d}} |\tilde{f}_{i,\theta}(z-x-y+c_i)(\eta_{\theta_0})^\vee(z)| dz \right)^2 \\ &\lesssim \int_{\mathbb{R}^{2d}} |\tilde{f}_{i,\theta}(z-x+c_i)|^2 |(\eta_{\theta_0})^\vee(z+y)| dz \\ &\lesssim N^{-3d/2} (|\tilde{f}_{i,\theta}|^2 * \chi_{\theta_0^*})(-x+c_i) \end{aligned}$$

for any $y \in [0, cN^{1/2}]^{2d}$ and $x \in \mathbb{R}^{2d}$. Integrating this in y variable, we conclude

$$\|f_i * \Xi_\theta\|_{L^2(B_{cN^{1/2}}(x+c_i))}^2 \lesssim N^{-d/2} (|\tilde{f}_{i,\theta}|^2 * \chi_{\theta_0^*})(-x).$$

Hence, we have

$$\int_{B_N} \prod_{i=1}^2 \left(\sum_{\theta \in \mathcal{P}_{N-1}} \|f_i * \Xi_\theta\|_{L^2(B_{cN^{1/2}}(x+c_i))}^2 \right)^{\frac{p}{4}} dx \lesssim N^{-\frac{dp}{4}} \int_{B_N} \prod_{i=1}^2 \left(\sum_{\theta \in \mathcal{P}_{N-1}} |\tilde{f}_{i,\theta}|^2 * \chi_{\theta_0^*}(x) \right)^{\frac{p}{4}} dx.$$

To apply Corollary 2.4, which is the bilinear Kakeya inequality, we use the change of variables: $y = N^{-1/2}x$. Then the above term is bounded by

$$\begin{aligned} &\lesssim N^{d(1-\frac{p}{4})} \int_{\mathbb{R}^{2d}} \left(\prod_{i=1}^2 \sum_{\theta} |\tilde{f}_{i,\theta}|^2 * \chi_{\theta_0^*}(N^{1/2}y) \right)^{\frac{p}{4}} dy \\ &\lesssim_\nu N^{d(1-\frac{p}{4})} \left(\prod_{i=1}^2 \int_{\mathbb{R}^{2d}} \sum_{\theta} |\tilde{f}_{i,\theta}|^2(N^{1/2}z) N^d dz \right)^{\frac{p}{4}} \lesssim N^{d(1-\frac{p}{4})} \prod_{i=1}^2 \|f_i\|_{L^2(\mathbb{R}^{2d})}^{\frac{p}{2}}. \end{aligned}$$

The last inequality follows from Plancherel's theorem and the pointwise comparability of $|\widehat{(f_i * \Xi_\theta)}^\vee|$ and $|\widehat{f_i * \Xi_\theta}|$. Hence, we obtain the desired result when $p \geq 4$.

The case that $2 < p < 4$ follows by interpolating two points $p = 2$ and $p = 4$ via Hölder's inequality. This completes the proof of Proposition 7.2. \square

Since we are interested in the decoupling, we have to change L^2 norm on the right hand side in Proposition 7.2 into $L^{p,N}$ norm. However, for large p , the exponent of N is too large to obtain the desired results if we simply apply Hölder's inequality to $L^{2,N}$ norm to obtain $L^{p,N}$ norm. As a compromise, we use a half and half mix of L^2 norm and $L^{p,N}$ norm.

Proposition 7.3. Fix $N \geq 1$. If $\text{supp } \widehat{f_i} \subset \mathcal{N}_{N^{-1}}([0,1]^d)$ and the Fourier supports of f_i are ν -transverse, then for each $\epsilon > 0$, $s \geq 2$, we have

$$\int_{B_N} \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_{N^{\frac{1}{2}}}}^2(x))}^{\frac{s}{2}} dx \lesssim_{\epsilon,\nu} N^\epsilon \max(N^{\frac{d}{4}(3+\frac{s}{2})}, N^{\frac{ds}{4}}) \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_N+a_{i,N}}^2)}^{\frac{s}{4}} \prod_{i=1}^2 \|f_i\|_{L^{s,N}(w_{B_N+a_{i,N}}^2)}^{\frac{s}{4}}$$

for some point $a_{i,N}$ depending on a choice of f_i but not a center of B_N .

Proof. Take $p = \frac{2+s}{2} \geq 2$. We will prove a rather weak inequality first;

$$\int_{B_N} \prod_{i=1}^2 \|f_i\|_{L^2(B_{N^{\frac{1}{2}}}(x))}^{\frac{s}{2}} dx \lesssim_{\epsilon,\nu} N^\epsilon \max(N^{\frac{d}{4}(3+\frac{s}{2})}, N^{\frac{ds}{4}}) \prod_{i=1}^2 \|f_i\|_{L^2(\mathbb{R}^{2d})}^{\frac{s}{4}} \prod_{i=1}^2 \|f_i\|_{L^{s,N}(\mathbb{R}^{2d})}^{\frac{s}{4}}.$$

We apply Lemma 6.2 with $Q = B_{2N}$ to the functions f_i . Since the error functions g_i are much tiny compared to f_i , we can ignore these functions. For convenience, we reorder indices $[j, m]$ in Lemma 6.2 so that we can write $f_i = \sum_{l=1}^{O(N^\epsilon)} f_{i,l} + g$. Then we have

$$\int_{B_N} \prod_{i=1}^2 \|f_i\|_{L^2(B_{N^{1/2}}(x))}^{\frac{s}{2}} dx \lesssim_\epsilon N^\epsilon \max_{l_1, l_2} \int_{B_N} \prod_{i=1}^2 \|f_{i,l_i}\|_{L^2(B_{N^{1/2}}(x))}^{\frac{s}{2}} dx + \prod_{i=1}^2 \|f_i\|_{L^2(\mathbb{R}^{2d})}^{\frac{s}{4}} \prod_{i=1}^2 \|f_i\|_{L^{s,N}(\mathbb{R}^{2d})}^{\frac{s}{4}}.$$

By using Plancherel's theorem and a finitely overlapping property, we get

$$\begin{aligned} \int_{B_N} \prod_{i=1}^2 \|f_{i,l_i}\|_{L^2(B_{N^{1/2}}(x))}^{\frac{s}{2}} dx &\lesssim \int_{B_N} \prod_{i=1}^2 \|f_{i,l_i}\|_{L^2(B_{N^{1/2}}(x))}^{\frac{p}{2}} dx \prod_{i=1}^2 \sup_{x \in B_N} \|f_{i,l_i}\|_{L^2(B_{N^{1/2}}(x))}^{\frac{s-p}{2}} \\ &\lesssim N^{\frac{d(s-p)}{2}} \int_{B_N} \prod_{i=1}^2 \|f_{i,l_i}\|_{L^2(B_{N^{1/2}}(x))}^{\frac{p}{2}} dx \prod_{i=1}^2 \left(\sum_{\theta} \|f_{i,l_i} * \Xi_\theta\|_{L^\infty(\mathbb{R}^{2d})}^2 \right)^{\frac{s-p}{4}}. \end{aligned}$$

In the next step, we apply Proposition 7.2 and recover the original function f_i from f_{i,l_i} by using the inequality (6.2). Then we can bound the above term by

$$\begin{aligned} &\lesssim N^{\frac{d(s-p)}{2}} \max(N^{d(2-\frac{p}{2})}, N^{d(1-\frac{p}{4})}) \prod_{i=1}^2 \|f_{i,l_i}\|_{L^2(\mathbb{R}^{2d})}^{\frac{p}{2}} \prod_{i=1}^2 \left(\sum_{\theta} \|f_{i,l_i} * \Xi_{\theta}\|_{L^{\infty}(\mathbb{R}^{2d})}^2 \right)^{\frac{s-p}{4}} \\ &\lesssim N^{\frac{d(s-p)}{2}} \max(N^{d(2-\frac{p}{2})}, N^{d(1-\frac{p}{4})}) \prod_{i=1}^2 \|f_i\|_{L^2(\mathbb{R}^{2d})}^{\frac{s}{4}} \left(\sum_{\theta} \|f_i * \Xi_{\theta}\|_{L^s(\mathbb{R}^{2d})}^2 \right)^{\frac{s}{8}} \\ &\lesssim N^{\frac{d(s-p)}{2}} \max(N^{d(2-\frac{p}{2})}, N^{d(1-\frac{p}{4})}) N^{\frac{d}{8}(s-2)} \prod_{i=1}^2 \|f_i\|_{L^2(\mathbb{R}^{2d})}^{\frac{s}{4}} \|f_i\|_{L^{s,N}(\mathbb{R}^{2d})}^{\frac{s}{4}}. \end{aligned}$$

The last inequality follows from Hölder's inequality. By direct computation, we can see that the exponent of N is $\max(\frac{d}{4}(3+\frac{s}{2}), \frac{ds}{4})$. Hence, we obtain the rather weak inequality. Next, by putting $f_i \eta_{B_{2N}}$ instead of f_i and using a finitely overlapping property, we get

$$\int_{B_N} \prod_{i=1}^2 \|f_i\|_{L^2(B_{N^{\frac{1}{2}}}(x))}^{\frac{s}{2}} dx \lesssim_{\epsilon, \nu} N^{\epsilon} \max(N^{\frac{d}{4}(3+\frac{s}{2})}, N^{\frac{ds}{4}}) \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_N}^2)}^{\frac{s}{4}} \prod_{i=1}^2 \|f_i\|_{L^{s,N}(w_{B_N}^2)}^{\frac{s}{4}}.$$

Observe that

$$w_{B_{N^{1/2}}(x)}^2(y) \lesssim \sum_{c_B \in N^{1/2}\mathbb{Z}^{2d}} 1_{B_{N^{1/2}}(x+c_B)}(y) w_{B_{N^{1/2}}(0)}^2(c_B).$$

By using this, we have

$$\begin{aligned} &\int_{B_N} \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_{N^{1/2}}(x)}^2)}^{\frac{s}{2}} dx \\ &\lesssim \sum_{B', B'' \in \mathcal{B}_{N^{\frac{1}{2}}}} w_{B_{N^{\frac{1}{2}}}}(c_{B'}) w_{B_{N^{\frac{1}{2}}}}(c_{B''}) \int_{B_N} \|f_1\|_{L^2(B_{N^{\frac{1}{2}}}(x+c_{B'}))}^{\frac{s}{2}} \|f_2\|_{L^2(B_{N^{\frac{1}{2}}}(x+c_{B''}))}^{\frac{s}{2}} dx \\ &\lesssim_{\epsilon, \nu} N^{\epsilon} \max(N^{\frac{d}{4}(3+\frac{s}{2})}, N^{\frac{ds}{4}}) \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_N}^2 + a_{i,N})}^{\frac{s}{4}} \|f_i\|_{L^{s,N}(w_{B_N+a_{i,N}}^2)}^{\frac{s}{4}} \end{aligned}$$

for some point $a_{i,N}$. This completes the proof of Proposition 7.3. \square

Now, we are ready to prove Theorem 7.1. Iterating Proposition 7.3 will lead to the desired inequality.

Proof of Theorem 7.1. Let $p \geq 2$. Fix a number $r = N^{2^{-M}} \geq 1$. By Bernstein's inequality,

$$\int_{B_N} |f_1 f_2|^{\frac{p}{2}} \lesssim \int_{B_{N+r}} \prod_{i=1}^2 \|f_i\|_{L^{\infty}(B_r(x))}^{\frac{p}{2}} dx \lesssim \int_{B_{N+r}} \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_r}^2)}^{\frac{p}{2}} dx.$$

We change the integrand into the average over cubes B_{r^2} ;

$$\lesssim r^{-2 \cdot 2d} \int_{B_{N+r+r^2}} \left(\int_{B_{r^2}(x)} \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_r}^2)}^{\frac{p}{2}} dy \right) dx.$$

Next, we apply Proposition 7.3 on each cube of sidelength r^2 . Then the above term is bounded by

$$\begin{aligned} &\lesssim r^{-2 \cdot 2d} r^{2 \cdot \max(\frac{d}{4}(3+\frac{p}{2}), \frac{dp}{2})} \int_{B_{N+r+r^2}} \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_{r^2}}^2(x+a_i))}^{\frac{p}{4}} \prod_{i=1}^2 \|f_i\|_{L^{p,r^2}(w_{B_{r^2}}^2(x+a_i))}^{\frac{p}{4}} dx \\ &\lesssim r^{-4d + \max(\frac{d}{2}(3+\frac{p}{2}), \frac{dp}{2})} \left(\int_{B_{N+r+r^2}} \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_{r^2}}^2(x+a_i))}^{\frac{p}{2}} dx \right)^{\frac{1}{2}} \prod_{i=1}^2 \left(\int_{B_{N+r+r^2}} \|f_i\|_{L^{p,r^2}(w_{B_{r^2}}^2(x+a_i))}^p dx \right)^{\frac{1}{4}} \end{aligned}$$

for some $a_i \in \mathbb{R}^{2d}$. The second inequality follows from Hölder's inequality. We will iterate the first term later. Consider the second term. By Fubini's theorem,

$$\int_{B_{N+r+r^2}} \sum_{\theta \in \mathcal{P}_{r^{-2}}} \|f_i * \Xi_{\theta}\|_{L^p(w_{B_{r^2}}^2(x+a_i))}^p dx \lesssim r^{4d} \|f_i\|_{L^{p,r^2}(\mathbb{R}^{2d})}^p.$$

By Proposition 5.3, we obtain

$$\begin{aligned} \|f_i\|_{L^{p,r^2}(\mathbb{R}^{2d})} &\lesssim D\left(\frac{N}{r^2}, p\right) \|f_i\|_{L^{p,N}(\mathbb{R}^{2d})} \\ &\lesssim_\epsilon \left(\frac{N}{r^2}\right)^{(\gamma_{lin}+\epsilon)} \|f_i\|_{L^{p,N}(\mathbb{R}^{2d})} = N^{(\gamma_{lin}+\epsilon)(1-\frac{2}{2M})} \|f_i\|_{L^{p,N}(\mathbb{R}^{2d})}. \end{aligned}$$

By using these two inequalities, we get

$$\int_{B_N} |f_1 f_2|^{\frac{p}{2}} \lesssim_\epsilon r^{-2d+\max(\frac{d}{2}(3+\frac{p}{2}), \frac{dp}{2})} N^{p(\gamma_{lin}+\epsilon)(\frac{1}{2}-\frac{1}{2M})} \left(\int_{B_{N+r+r^2}} \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_{r^2}(x+a_i)}^2)}^{\frac{p}{2}} dx \right)^{\frac{1}{2}} \prod_{i=1}^2 \|f_i\|_{L^{p,N}(\mathbb{R}^{2d})}^{\frac{p}{4}}.$$

Repeating this process again on the first term gives

$$\begin{aligned} &\int_{B_{N+r+r^2}} \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_{r^2}(x+a_i)}^2)}^{\frac{p}{2}} dx \\ &\lesssim_\epsilon r^{2 \cdot (-2d+\max(\frac{d}{2}(3+\frac{p}{2}), \frac{dp}{2}))} N^{p(\gamma_{lin}+\epsilon)(\frac{1}{2}-\frac{2}{2M})} \prod_{i=1}^2 \left(\int_{B_{N+r+r^2+r^4}} \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_{r^4}(x+b_i)}^2)}^{\frac{p}{2}} dx \right)^{\frac{1}{2}} \|f_i\|_{L^{p,N}(\mathbb{R}^{2d})}^{\frac{p}{4}} \end{aligned}$$

for some $b_i \in \mathbb{R}^{2d}$. Combining these two inequalities gives

$$\begin{aligned} &\int_{B_N} |f_1 f_2|^{\frac{p}{2}} \\ &\leq C_{p,\epsilon} r^{2 \cdot (-2d+\max(\frac{d}{2}(3+\frac{p}{2}), \frac{dp}{2}))} N^{p(\gamma_{lin}+\epsilon)(\frac{3}{4}-\frac{2}{2M})} \left(\int_{B_{N+r+r^2+r^4}} \prod_{i=1}^2 \|f_i\|_{L^2(w_{B_{r^4}(x+b_i)}^2)}^{\frac{p}{2}} dx \right)^{\frac{1}{4}} \prod_{i=1}^2 \|f_i\|_{L^{p,N}(\mathbb{R}^{2d})}^{\frac{p}{4}(1+\frac{1}{2})}. \end{aligned}$$

By repeating this process $M-2$ times more, recalling that $r = N^{1/2^M}$ and using Hölder's inequality, we obtain

$$\begin{aligned} &\int_{B_N} |f_1 f_2|^{\frac{p}{2}} \\ &\leq C_{p,\epsilon}^M r^{C'} r^{M(-2d+\max(\frac{d}{2}(3+\frac{p}{2}), \frac{dp}{2}))} N^{p(\gamma_{lin}+\epsilon)(1-\frac{M}{2M})} \prod_{i=1}^2 \|f_i\|_{L^2(\mathbb{R}^{2d})}^{\frac{p}{2M+1}} \prod_{i=1}^2 \|f_i\|_{L^{p,N}(\mathbb{R}^{2d})}^{\frac{p}{2}(1-\frac{1}{2M})} \\ &\leq N^{\frac{M \log C_{p,\epsilon}}{\log N}} N^{\frac{C''}{2M}} N^{\frac{(-2d+\max(\frac{d}{2}(3+\frac{p}{2}), \frac{dp}{2}))M}{2M}} N^{p(\gamma_{lin}+\epsilon)(1-\frac{M}{2M})} \prod_{i=1}^2 \|f_i\|_{L^{p,N}(\mathbb{R}^{2d})}^{\frac{p}{2}}. \end{aligned}$$

By using a standard localization argument and summing over cubes B_N and raising to the power $1/p$, we have

$$\| |f_1 f_2|^{\frac{1}{2}} \|_{L^p(\mathbb{R}^{2d})} \leq N^{\frac{M \log C_{p,\epsilon}}{p \log N} + \frac{C'}{2M} + \frac{(-2d+\max(\frac{d}{2}(3+\frac{p}{2}), \frac{dp}{2}))M}{2Mp} + (\gamma_{lin}+\epsilon)(1-\frac{M}{2M})} \prod_{i=1}^2 \|f_i\|_{L^{p,N}(\mathbb{R}^{2d})}^{1/2}.$$

By the definition of γ_{bil} , Proposition 5.1 and 5.2, we have

$$N^{\gamma_{bil}} \lesssim_\epsilon N^{\frac{M \log C_{p,\epsilon}}{p \log N} + \frac{C'}{2M} + \frac{(-2d+\max(\frac{d}{2}(3+\frac{p}{2}), \frac{dp}{2}))M}{2Mp} + (\gamma_{lin}+\epsilon)(1-\frac{M}{2M})},$$

using the assumption (4.1) and rearranging this inequality,

$$N^{\gamma_{bil}(p_0) \frac{M}{2M}} \lesssim_\epsilon N^{\frac{M \log C_{p_0,\epsilon}}{p_0 \log N} + \frac{C'}{2M} + \frac{M}{2M} \cdot \max(\frac{d}{2}(\frac{1}{2}-\frac{1}{p_0}), \frac{d}{2}(1-\frac{4}{p_0})) + \epsilon}.$$

By taking N and M sufficiently large, we obtain

$$\gamma_{bil}(p_0) \leq \max\left(\frac{d}{2}\left(\frac{1}{2}-\frac{1}{p_0}\right), \frac{d}{2}\left(1-\frac{4}{p_0}\right)\right).$$

This completes the proof. \square

8. PROOF OF COROLLARY 1.2

To deduce Corollary 1.2, we utilize the l^p decoupling for the truncated hyperbolic paraboloid, which was established by Bourgain and Demeter.

Let $n \geq 2$ be an integer. For each $v = (v_1, \dots, v_{n-1}) \in (\mathbb{R} \setminus 0)^{n-1}$, the truncated hyperbolic paraboloid corresponding to v is defined by

$$H_v^{n-1} = \{(\xi_1, \dots, \xi_{n-1}, v_1 \xi_1^2 + \dots + v_{n-1} \xi_{n-1}^2) : (\xi_1, \dots, \xi_{n-1}) \in [0, 1]^{n-1}\}.$$

As Section 5, for $\delta > 0$ and $Q \subset [0, 1]^{n-1}$ we define the δ -neighborhood of H_v^{n-1} above Q to be

$$\mathcal{N}_\delta(Q) = \{(\xi_1, \dots, \xi_{n-1}, v_1 \xi_1^2 + \dots + v_{n-1} \xi_{n-1}^2 + t) : (\xi_1, \dots, \xi_{n-1}) \in Q, -\delta \leq t \leq \delta\}.$$

Theorem 8.1. [4] Let $n \geq 2$. If $\text{supp}(\hat{f}) \subset \mathcal{N}_\delta([0, 1]^{n-1})$ then for $\epsilon > 0$ and for $l \geq 1$

$$\|f\|_{L^p(w_{B_N}^l)} \lesssim_{\epsilon, l} \delta^{-\epsilon} \max(\delta^{-\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})}, \delta^{\frac{n}{p}-\frac{n-1}{2}}) \left(\sum_{Q \in \mathcal{P}_\delta} \|f_{\mathcal{N}_\delta(Q)}\|_{L^p(w_{B_N}^l)}^p \right)^{\frac{1}{p}}.$$

As a corollary of Theorem 8.1, we can get decouplings for hypersurfaces satisfying some principal curvature conditions. The special case $j = 1$ in Corollary 8.2 is the same as Lemma 2.4 in [4], but the proof of Corollary 8.2 is identical to that of Lemma 2.4 in [4].

Corollary 8.2 (A lower dimensional decoupling). Fix integers j, m with $m \geq 2$ and $j \geq 0$. We also fix $v_1, \dots, v_{m-1} \in \{1, -1\}$ and let $|a_1|, \dots, |a_j| \lesssim 1$ be arbitrary, possibly zero. For each $Q \subset [0, 1]^{m-1+j}$, we denote by $\mathcal{N}_\delta(Q)$ the δ -neighborhood of $H_{(v_1, \dots, v_{m-1}, a_1, \dots, a_j)}^{m-1+j}$ above Q .

For each $p \geq 2$ and $f : \mathbb{R}^{m+j} \rightarrow \mathbb{C}$ with $\text{supp}(\hat{f}) \subset \mathcal{N}_\delta([0, 1]^{m-1+j})$ we have

$$\|f\|_{L^p(w_{B_N})} \lesssim_{\epsilon} \delta^{-\frac{j}{2}+\frac{j}{p}-\epsilon} \max(\delta^{-\frac{m-1}{2}(\frac{1}{2}-\frac{1}{p})}, \delta^{\frac{m}{p}-\frac{m-1}{2}}) \left(\sum_{Q \in \mathcal{P}_\delta} \|f_{\mathcal{N}_\delta(Q)}\|_{L^p(w_{B_N})}^p \right)^{\frac{1}{p}}.$$

Proof. We may assume that $B_N = [0, N]^{2d}$. By applying the trivial l^p decoupling in the directions of e_m, \dots, e_{m+j-1} and Lemma 3.1, it suffices to prove that for each $\alpha = (\alpha_1, \dots, \alpha_j) \in [0, 1]^j$

$$\left\| \sum_{Q \in \mathcal{P}_{\delta, \alpha}} f_{\mathcal{N}_\delta(Q)} \right\|_{L^p(B_N)}^p \lesssim_{\epsilon} \delta^{-\epsilon} \max(\delta^{-\frac{m-1}{2}(\frac{p}{2}-1)}, \delta^{m-\frac{p(m-1)}{2}}) \sum_{Q \in \mathcal{P}_{\delta, \alpha}} \|f_{\mathcal{N}_\delta(Q)}\|_{L^p(w_{B_N})}^p,$$

where $\mathcal{P}_{\delta, \alpha}$ consists of cubes $Q \in \mathcal{P}_\delta$ satisfying that $\mathcal{N}_\delta(Q)$ intersects the paraboloid

$$\{(\xi_1, \dots, \xi_{m-1}, \alpha_1, \dots, \alpha_j, \xi_{m+j}) : \xi_{m+j} = \xi_1^2 + \dots + \xi_{m-1}^2 + \alpha_1 \alpha_1^2 + \dots + \alpha_j \alpha_j^2\}.$$

By using linear transformations, we may assume that $\alpha_1 = \dots = \alpha_j = 0$. Now, for each cube $Q \in \mathcal{P}_{\delta, 0}$, the set $\mathcal{N}_\delta(Q)$ is contained in the $O(\delta)$ -neighborhood of the cylinder

$$\{(\xi_1, \dots, \xi_{m+j}) : \xi_{m+j} = \xi_1^2 + \dots + \xi_{m-1}^2\}.$$

We apply Theorem 8.1 with $n = m$. Then integrating over $[0, N]$ on x_m, \dots, x_{m+j-1} variables gives the desired results. \square

Let $E'f$ be the extension operator associated with $H_{(v_1, \dots, v_{m-1}, a_1, \dots, a_j)}^{m-1+j}$. One of the equivalent formulations of Corollary 8.2 is that for $f : [0, 1]^{m+j-1} \rightarrow \mathbb{C}$ and $\delta, \epsilon > 0$

$$(8.1) \quad \|E'_{[0, 1]^{m-1+j}} f\|_{L^p(w_{B_N})} \lesssim_{\epsilon} \delta^{-\frac{j}{2}+\frac{j}{p}-\epsilon} \max(\delta^{-\frac{m-1}{2}(\frac{1}{2}-\frac{1}{p})}, \delta^{\frac{m}{p}-\frac{m-1}{2}}) \left(\sum_{Q \in \mathcal{P}_\delta} \|E'_Q f\|_{L^p(w_{B_N})}^p \right)^{\frac{1}{p}}.$$

We also utilize Cauchy's interlacing theorem, which gives the relationship between the eigenvalues of a matrix and the eigenvalues of its principal submatrices.

Theorem 8.3 (Interlacing theorem). Fix positive integers n, m with $m \leq n-1$. Let A be a Hermitian $n \times n$ matrix, and let B be a $m \times m$ principal submatrix of A . If the eigenvalues of A are $\lambda_1(A), \dots, \lambda_n(A)$ with $\lambda_1(A) \leq \dots \leq \lambda_n(A)$, and those of B are $\lambda_1(B), \dots, \lambda_m(B)$ with $\lambda_1(B) \leq \dots \leq \lambda_m(B)$, then for $1 \leq k \leq m$

$$\lambda_k(A) \leq \lambda_k(B) \leq \lambda_{k+n-m}(A).$$

Proof of Corollary 1.2. We may assume that $B_N = [0, N]^{2d}$. By using a linear transformation, we can assume that $\sup_i \sup_{\xi \in [-5d, 5d]^d} |\Phi_i(\xi)| \leq 1$. Suppose that L is a hyperplane in \mathbb{R}^d and R is a rotation satisfying that $R^{-1}(L) = \{0\} \times \mathbb{R}^{d-1}$. Let $g : [0, 1]^{d-1} \rightarrow \mathbb{C}$ be a function. By Theorem 1.1 and a change of variables, it suffices to show that

$$\|E_{S|L} g\|_{L^p(w_{B_N}^4)} \lesssim_\epsilon N^\epsilon \max(N^{\frac{d+k-1}{2}(\frac{1}{2}-\frac{1}{p})}, N^{\frac{d-1}{2}-\frac{d}{p}}) \left(\sum_{\alpha \in \mathcal{P}_{N-1}} \|E_\alpha g\|_{L^p(w_{B_N}^4)}^p \right)^{\frac{1}{p}},$$

where $S|_L = \{(\xi_2, \dots, \xi_d, \Phi(R(0, \xi_2, \dots, \xi_d)))\}$.

We claim that there exists $\epsilon_1 > 0$ such that

$$(8.2) \quad \inf_{L'} \sup_i \sup_{(\xi_1, \dots, \xi_d) \in L'} |\Phi_i(\xi_1, \dots, \xi_d)| > \epsilon_1$$

where the infimum of L' runs over all $(k+1)$ -dimensional subspaces L' containing the origin. For a contradiction, suppose that (8.2) does not hold. Since the Grassmannian is sequentially compact, we can take a plane L' such that all phase functions vanish on the plane L' . However, the assumption that S is of type k leads to a contradiction. Hence, the inequality (8.2) holds true. Note that (8.2) still holds even if the infimum of L' runs over all j -dimensional subspaces L' containing the origin for $k+1 \leq j \leq d$.

Let $j_0 = 1$. Since $k \leq d-2$, by using (8.2), a linear transformation and the spectral theorem, we can write that

$$(\Phi_1 \circ R \circ R_1)(0, \xi_2, \dots, \xi_d) = a_{1,2}\xi_2^2 + \dots + a_{1,d}\xi_d^2,$$

where $|a_{1,2}| \geq \epsilon_1/10d$ and $|a_{1,2}| \geq \dots \geq |a_{1,d}|$ and R_1 is some rotation matrix fixing ξ_1 -axis. Let t_1 be the smallest positive integer such that $|a_{1,s}| \notin [(\frac{\epsilon_1}{10d})^{t_1+10^{10d}}, 10^d(\frac{\epsilon_1}{10d})^{t_1}]$ for any s . Let j_1 be the largest integer such that $|a_{1,j_1}| > 10^d(\frac{\epsilon_1}{10d})^{t_1}$. If $j_1 \geq d-k$, we stop this process. Otherwise, by applying the spectral theorem to $\xi_{j_1+1}, \dots, \xi_d$ variables and using (8.2) again, we can write that

$$(\Phi_2 \circ R \circ R_1 \circ R_2)(0, \dots, 0, \xi_{j_1+1}, \dots, \xi_d) = a_{2,j_1+1}\xi_{j_1+1}^2 + \dots + a_{2,d}\xi_d^2,$$

where R_2 is some rotation fixing ξ_1 -axis, \dots , ξ_{j_1} -axis. Here, the constants satisfy $|a_{2,j_1+1}| \geq \dots \geq |a_{2,d}|$ and $|a_{2,j_1+1}| \geq \epsilon_1/10d$. Let t_2 be the smallest positive integer such that $|a_{2,s}| \notin [(\frac{\epsilon_1}{10d})^{t_2+10^{10d-1}}, 10^d(\frac{\epsilon_1}{10d})^{t_2}]$. We define j_2 as before and repeat this process at most $(d-k-3)$ times more.

To summarize, there exists an integer l with $1 \leq l \leq d-k-1$ such that for any $s = 1, \dots, l$

$$(\Phi_s \circ R \circ \tilde{R})(0, \dots, 0, \xi_{j_{s-1}+1}, \dots, \xi_d) = \sum_{m=j_{s-1}+1}^{j_s} a_{s,m}\xi_m^2 + \sum_{u,v=j_s+1}^d b_{s,u,v}\xi_u\xi_v,$$

where $|a_{s,m}| \geq 10^d(\epsilon_1/10d)^{t_s}$, $|b_{s,u,v}| \leq (\epsilon_1/10d)^{t_s+10^{10d-s+1}}$ and $\tilde{R} = R_1 \circ \dots \circ R_l$. Moreover,

$$t_s \leq d + d \cdot 10^{10d-s+1}, \quad j_{s-1} + 1 \leq j_s$$

for $s = 1, \dots, l$. Note that $j_l \geq d-k$, and \tilde{R} fixes ξ_1 -axis.

Claim. There exists ϵ_2 such that

$$D = \left| \det \begin{bmatrix} \frac{\partial G}{\partial \xi_2 \partial \xi_2} & \cdots & \frac{\partial G}{\partial \xi_2 \partial \xi_{j_l}} \\ \vdots & \ddots & \vdots \\ \frac{\partial G}{\partial \xi_{j_l} \partial \xi_2} & \cdots & \frac{\partial G}{\partial \xi_{j_l} \partial \xi_{j_l}} \end{bmatrix} \right| \geq \epsilon_2 > 0,$$

where $G(\xi) = \Phi_1(R\xi) + c_2\Phi_2(R\xi) + \dots + c_l\Phi_l(R\xi)$ for some $c_2, \dots, c_l \lesssim 1$. Here, the constant ϵ_2 is independent of the choice of L .

For the moment, we assume that the claim holds. We define the following linear transformation:

$$T : (x_1, \dots, x_{2d-1}) \mapsto (x_1, \dots, x_d, x_{d+1} + c_2x_d, \dots, x_{d+l-1} + c_lx_d, x_{d+l+1}, \dots, x_{2d-1}).$$

By the claim, Cauchy's interlacing theorem and the spectral theorem, we can take a rotation R' so that $E_{S|L}g(Tx) = (E_{S''}(g \circ R'))(x)$, where

$$S'' = \{(\xi_2, \dots, \xi_d, a_2\xi_2^2 + \dots + a_d\xi_d^2, \dots)\},$$

and $|a_2|, \dots, |a_{d-k}| \geq \epsilon_3$ for some $\epsilon_3 > 0$, and $|a_2|, \dots, |a_d| \lesssim 1$. For fixed $x_{d+1}, \dots, x_{2d-1} \in [0, N]$, we apply (8.1) with $m-1 = d-k-1$, $j = k$. Next, by integrating over $[0, N]$ on x_{d+1}, \dots, x_{2d-1}

$$\|E_{S|L}g\|_{L^p(B_N)}^p \lesssim_\epsilon N^\epsilon \max(N^{\frac{d+k-1}{2}(\frac{p}{2}-1)}, N^{\frac{(d-1)p}{2}-d}) \sum_{\alpha \in \mathcal{P}_{N-1}} \|E_\alpha g\|_{L^p(w_{B_N}^4)}^p.$$

Now, Lemma 3.1 gives the desired results. Therefore, it suffices to prove the claim.

Proof of the claim. Take

$$c_2 = \left(\frac{\epsilon_1}{10d}\right)^{t_1 + \sum_{i=2}^l 2^{i-2}t_i + 1}, \quad c_l = \left(\frac{\epsilon_1}{10d}\right)^{t_1 + \sum_{i=2}^{l-1} 2^{i-1}t_i + (2^{l-1}-1)t_l + \frac{3}{2}}.$$

Note that $c_2^2(\frac{\epsilon_1}{10d})^{-\frac{1}{2}-t_1} = c_l(\frac{\epsilon_1}{10d})^{t_l}$. Next, we define

$$c_i = (c_l c_{i-1} (\frac{\epsilon_1}{10d})^{t_i + t_{i-1}})^{1/2}$$

for $i = 3, \dots, l-1$. Note that $c_l \leq \dots \leq c_2$. One can obtain

$$D = \left| \prod_{i=2}^{j_1} (a_{1,i} + O(c_2)) \right| \prod_{r=2}^l \prod_{i=1}^{j_r - j_{r-1}} (a_{r, j_{r-1} + i} c_r + o(a_{r, j_{r-1} + i} c_r))$$

via routine computations. Since we have the uniform lower bound of $|a_{s,m}|$, this completes the proof of the claim. \square

REFERENCES

- [1] Bennett, J.: Aspects of multilinear harmonic analysis related to transversality. Harmonic analysis and partial differential equations, 1–28, Contemp. Math., 612, Amer. Math. Soc., Providence, RI, 2014.
- [2] Bennett, J., Carbery, A., Tao, T.: On the multilinear restriction and Kakeya conjectures. Acta Math. **196**(2), 261–302 (2006).
- [3] Bourgain, J.: Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces. Israel J. Math. **193**(1), 441–458 (2013).
- [4] Bourgain, J., Demeter, C.: Decouplings for curves and hypersurfaces with nonzero Gaussian curvature. Preprint arXiv:1409.1634 (2015).
- [5] Bourgain, J., Demeter, C.: Decouplings for surfaces in \mathbb{R}^4 . J. Funct. Anal. **270**(4), 1299–1318 (2016).
- [6] Bourgain, J., Demeter, C.: Mean value estimates for Weyl sums in two dimensions. Preprint arXiv:1509.05388 (2015).
- [7] Bourgain, J., Demeter, C.: The proof of the l^2 decoupling conjecture. Annals of Math. **182**(1), 351–389 (2015).
- [8] Bourgain, J., Demeter, C.: A study guide for the l^2 decoupling theorem. Preprint arXiv:1604.06032 (2016).
- [9] Bourgain, J., Demeter, C., Shao, G.: Sharp bounds for the cubic Parsell-Vinogradov system in two dimensions. Preprint arXiv:1608.06346 (2016).
- [10] Bourgain, J., Demeter, C., Guth, L.: Proof of the main conjecture in Vinogradov’s mean value theorem for degrees higher than three. Preprint arXiv:1512.01565 (2016).
- [11] Bourgain, J., Guth, L.: Bounds on oscillatory integral operators based on multilinear estimates. Geom. Funct. Anal. **21**(6), 1239–1295 (2011).
- [12] Demeter, C.: Incidence theory and discrete restriction estimates. Preprint arXiv:1401.1873 (2014).
- [13] Garrigós, G., Schlag, W., Seeger, A.: Improvements in Wolff’s inequality for decompositions of cone multipliers. <http://webs.um.es/gustavo.garrigos/papers/GSS7bis.pdf>.
- [14] Garrigós, G., Seeger, A.: A mixed norm variant of Wolff’s inequality for paraboloids. Harmonic Analysis and Partial Differential Equations, Contemporary Mathematics, vol. 505, Amer. Math. Soc., Providence, RI, 2010, pp. 179–197.
- [15] Garrigós, G., Seeger, A.: On plate decompositions of cone multipliers. Proc. Edinb. Math. Soc. **52**(03), 631–651 (2009).
- [16] Guth, L.: Mini-course Notes on Decoupling and multilinear estimates in harmonic analysis.
- [17] Laba, I., Pramanik, M.: Wolff’s inequality for hypersurfaces. Collect. Math. Extra(Vol. Extra), 293–326 (2006).
- [18] Laba, I., Wolff, T.: A local smoothing estimate in higher dimensions. J. Anal. Math. **88**(1), 149–171 (2002).
- [19] Tao, T., Vargas, A., Vega, L.: A bilinear approach to the restriction and Kakeya conjectures. J. Amer. Math. Soc. **11**(4), 967–1000 (1998).
- [20] Wolff, T.: Local smoothing type estimates on L^p for large p . Geom. Funct. Anal. **10**(5), 1237–1288 (2000).

DEPARTMENT OF MATHEMATICS, POHANG UNIVERSITY OF SCIENCE AND TECHNOLOGY, POHANG 790-784, REPUBLIC OF KOREA
E-mail address: ock9082@postech.ac.kr